ORTHOGONAL QUANTUM GROUP INVARIANTS OF LINKS

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ABSTRACT. We study the Chern-Simons partition function of orthogonal quantum group invariants, and propose a new orthogonal Labastida-Mariño-Ooguri-Vafa conjecture as well as degree conjecture for free energy associated to the orthogonal Chern-Simons partition function. We prove the degree conjecture and some interesting cases of orthogonal LMOV conjecture. In particular, We provide a formula of colored Kauffman polynomials for torus knots and links, and applied this formula to verify certain case of the conjecture at roots of unity except 1. We also derive formulas of Lickorish-Millett type for Kauffman polynomials and relate all these to the orthogonal LMOV conjecture.

1. INTRODUCTION

1.1. **Overview.** Jones' seminal papers [18, 19] initiated a new era in knot theory. The HOMFLY polynomial [13] and Kauffman [23] polynomial for links were subsequently discovered. In the 1990's, Witten-Reshetikhin-Turaev constructed the colored version of these invariants, either by path integrals in physics [54], or by the representation theory of quantum groups [44, 45]. These works lead to a unified understanding of quantum group invariants of links.

The colored HOMFLY polynomials, which are associated to the special linear quantum groups, have been studied more carefully after physicists proposed a conjectural relationship between Chern-Simons theory and Gromov-Witten invariants. The Mariño-Vafa formula and the topological vertex [1, 27, 30, 31] are examples illustrating this so-called string duality. The Labastida-Mariño-Ooguri-Vafa conjecture [25, 26, 36] gave highly non-trivial relations between colored HOMFLY polynomials. The first such relation is the classical Lichorish-Millett theorem [28]. The integers coefficients that appear in the LMOV conjecture are called the BPS numbers in string theory, and also related to the integrality in the Gopakumar-Vafa conjecture [16] for Gromov-Witten invariants [39]. By using the cabling technique, Xiao-Song Lin and Hao Zheng [29] obtained a formula for colored HOMFLY polynomials of torus links in terms of Littlewood-Richardson coefficients, and they were able to check certain cases of the LMOV conjecture for a few (small) torus knots and links. The LMOV conjecture was recently proved by Kefeng Liu and Pan Peng [32], based on the cabling technique and a careful degree analysis of the cut-join equations.

Actually the LMOV conjecture is part of a bigger picture, the large N duality, proposed by 't Hooft [49] in the 1970's. Large N duality states that the duality between Chern-Simons gauge theory of S^3 and topological string theory on the resolved conifold.

In mathematics, the LMOV conjecture predicts that the reformulated invariants (some combination) of colored HOMFLY/Kauffman polynomials are in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$, where q is the quantum deformation number. Through this way, these reformulated invariants has the similar expression as the original HOMFLY/Kauffman polynomials which has variables $q-q^{-1}$, t and t^{-1}.

1.2. **Orthogonal Labastida-Mariño-Ooguri-Vafa Conjecture.** The study of colored Kauffman polynomials is more difficult. For instance, the definition of the Chern-Simons partition function for the orthogonal quantum groups involves representations of the Brauer centralizer

algebras, which admit more complicated orthogonal relations (see [40, 41, 42]). The orthogonal analog of cut-join equation [30, 32] can be found in [10].

In this paper, we propose a new conjecture on the reformulated invariants, developed in collaboration with Nicolai Reshetikhin, which is the orthogonal quantum group analog of the original LMOV conjecture. Let $\mathcal{L}$ be a link with L components and let $Z_{CS}^{SO}(\mathcal{L}, q, t)$ be the orthogonal Chern-Simons partition function defined in Section 4. Expand the free energy

$$F^{SO}(\mathcal{L}, q, t) = \log Z_{CS}^{SO}(\mathcal{L}, q, t) = \sum_{\overrightarrow{\mu} \neq \overrightarrow{0}} F_{\overrightarrow{\mu}}^{SO} pb_{\overrightarrow{\mu}}(\overrightarrow{z}).$$

Then the reformulated invariants are defined by

$$g_{\overrightarrow{\mu}}(q, t) = \sum_{k | \overrightarrow{\mu}} \frac{\mu(k)}{k} F_{\overrightarrow{\mu}/k}^{SO}(q^k, t^k).$$

We conjecture that

Conjecture 1.1 (Orthogonal LMOV)**.**

$$\frac{\mathfrak{z}_{\overrightarrow{\mu}}(q - q^{-1})^2 \cdot [g_{\overrightarrow{\mu}}(q, t) - g_{\overrightarrow{\mu}}(q, -t)]}{2 \prod_{\alpha=1}^{L} \prod_{i=1}^{\ell(\mu^\alpha)} (q^{\mu_i^\alpha} - q^{-\mu_i^\alpha})} \in \mathbb{Z}[q - q^{-1}][t, t^{-1}].$$

and

Conjecture 1.2 (Degree)**.**

$$val_u(F_{\overrightarrow{\mu}}^{SO}) \geq \ell(\overrightarrow{\mu}) - 2,$$

where $q = e^u$, $val_u(F_{\overrightarrow{\mu}}^{SO})$ is the valuation of the variable u and $\ell(\overrightarrow{\mu})$ is the sum of the lengths of the partition corresponding to each component of the link $\mathcal{L}$.

This conjecture is a mathematical formulation of the conjecture made by Bouchard-Florea-Mariño [7], and the integer coefficients on the right hand side of the above conjecture is closely related to BPS numbers in string theory [7]. More recent progress can be found in [34], which is a refined version of [7]. The framing version can be found in [6, 38]. Our formulation is still quite different from that in [7, 34]. The reason for this is that [7, 34] uses representations of Hecke algebra, whereas our approach is based on representations of the Birman-Murakami-Wenzl algebra, and uses type-B Schur function instead of type-A Schur function as the basis in the orthogonal Chern-Simons partition function.

Theorems that partly answer the orthogonal LMOV conjecture proposed in this paper are listed below. For more precise statements of these theorems, see Sections 5, 7, 8 and 9.

Theorem 1.3. *The conjecture is true for all partitions when the link is trivial (namely is a disjoint union of unlinked unknots).*

Theorem 1.4. *The conjecture is true for partitions of the shape $\overrightarrow{\mu} = ((1^{d_1}), (1^{d_2}), \cdots, (1^{d_L}))$, where $(1^{d_\alpha}) = (1, 1, ..., 1) \vdash d_\alpha$ for $1 \leq \alpha \leq L$.*

Theorem 1.5. *The conjecture is true if and only if it is true for partitions of the shape $\overrightarrow{\mu} = ((d_1), (d_2), \cdots, (d_L))$.*

Theorem 1.6. *The conjecture asymptotically holds (for all partitions $\overrightarrow{\mu}$ and all knots/links) as q tends to 1.*

Theorem 1.7. *The conjecture is true when $\mathcal{L}$ is: the torus knots/links $T(2,k)$, where k is odd/even, and each component of the partition $\overrightarrow{\mu}$ is of the form (1), $(1,1)$ or (2); the two components torus link $T(2,2k)$ for partition $(3),(1)$; the three components torus link $T(3,3k)$ for the partition $(2),(1),(1)$. These examples give evidence for the conjecture at non-trivial roots of unity.*

We also prove the degree conjecture.

Theorem 1.8. *The following degree estimation holds*

$$val_u(F^{SO}_{\overrightarrow{\mu}}) \geq \ell(\overrightarrow{\mu}) - 2.$$

In addition, we use the cabling technique developed in [29] to calculate colored Kauffman polynomials for torus knots and links, which are employed to test the orthogonal LMOV conjecture (Theorem 1.7).

This paper is organized as follows: In Section 2, we review some basic knowledge of partitions, the Birman-Murakami-Wenzl algebra and irreducible representation of the Brauer algebra. In Section 3, we review the definition of the quantum group invariants of links and use the cabling formula to simplify the computation of these invariants. As an application of the cabling formula, we obtain colored Kauffman polynomials of all torus knots and links for all partitions (irreducible representations). In Section 4, we define the Chern-Simons partition function for orthogonal quantum groups and the corresponding reformulated invariants. Also, we compute the orthogonal Chern-Simons partition function for disjoint union of unknots (Theorem 1.3). In Section 5, we propose a new orthogonal LMOV conjecture and degree conjecture. Then we test torus knots and links as supporting examples (Theorem 1.7), which can not be treated as special cases of the proof in the following sections. In Section 6, we obtain formulas of Lickorish-Millett type by using skein relations at the intersections of two different link components. This trick is also widely used in Section 7. Anyway, this section is quite independent and such Lickorish-Millett type formulas can also be treated as an application of the orthogonal LMOV conjecture, which is the starting point of this paper. In Section 7, we prove the equivalence between the vanishing of the first three coefficients of $F_{\overrightarrow{\mu}}$ for trivial partitions $\overrightarrow{\mu}$ (each component of partitions have only one box), predicted by the degree conjecture, and the Lichorish-Millett type formulas obtained in Section 6. We also prove the orthogonal LMOV conjecture for column-like Young diagram (Theorem 1.4) as a generalization of such Lichorish-Millett type formulas. In Section 8 and 9, we prove that if the orthogonal LMOV conjecture is valid for the case of rows, then the orthogonal LMOV is valid for all partitions (Theorem 1.5). In addition, the proof of the degree conjecture is also presented there (Theorem 1.7), which implies that the orthogonal LMOV Conjecture asymptotically holds (for all partitions $\overrightarrow{\mu}$ and all knots/links) as q tends to 1 (Theorem 1.6). In Section 10 (Appendix), we first compute explicit expressions of the Chern-Simon partition function for the unknot. We then review an alternative definition of the colored Kauffman polynomial via the Markov trace (skein approach) and test the Hopf link for the orthogonal LMOV conjecture by using this new definition. We also give an explicit computation of the quantum trace for orthogonal quantum groups directly from the universal R-matrix. Finally, we list the character table of Brauer algebra and type-B Schur functions, whose specialization gives colored Kauffman polynomials of the unknot (quantum dimensions) for small partitions. These tables are mainly used to compute colored Kauffman polynomial for torus knots and links. The tables of the integers coefficients predicted by the orthogonal LMOV conjecture are also presented.

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2. Young Diagram and Birman-Murakami-Wenzl Algebra

2.1. **Partition and Young Diagram.** A *composition* μ of n, denoted by $\mu \models n$, is a finite sequence of positive integers $(\mu_1, \mu_2, ..., \mu_\ell)$ such that

$$\mu_1 + \mu_2 + ... + \mu_\ell = n.$$

The number of parts in μ is called the length of μ and denote by $\ell = \ell(\mu)$. The size of composition μ is defined by

$$|\mu| = \sum_{i=1}^{\ell(\lambda)} \mu_i.$$

A *partition* λ is a composition such that

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_\ell > 0.$$

Denote by $\mathcal{P}$ the set of all partitions. We identify a partition with its Young diagram.

If $|\lambda| = d$, we say λ is a partition of d, denote by $\lambda \vdash d$.

We use $m_i(\lambda)$ to denote the number of times that i appears in λ. Denote the automorphism group of the partition λ by $Aut(\lambda)$.

The order of $Aut(\lambda)$ is given by

$$|Aut(\lambda)| = \prod_i m_i(\lambda)!$$

There is another way to rewrite a partition λ in the following format

$$(1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdot \cdot\cdot)$$

For Instance, we have $(5, 3, 3, 2, 2, 2, 1) = (1^1 2^3 3^2 5^1)$

Define the following numbers associated to a partition λ.

$$\mathrm{z}_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!,$$

$$\kappa_\lambda = \prod_{j=1}^{\ell(\lambda)} \lambda_j(\lambda_j - 2j + 1).$$

2.2. **Partitionable set and infinite series.** Following the notations of [32], we present some basic knowledge of partitionable set here.

The concept of partition can be generalized to the following partitionable set.

A countable set $(S, +)$ is called a *partitionable* set if the following holds

(1) S is totally ordered.
(2) S is an Abelian semi-group with summation " $+$ ".
(3) The minimum element $\mathbf{0}$ in S is the zero-element of the semi-group, i.e., for any $a \in S$,

$$\mathbf{0} + a = a = a + \mathbf{0}.$$

For simplicity, we may briefly write S instead of $(S, +)$.

The following sets are examples of partitionable set:

(1) The set of all nonnegative integers $\mathbb{Z}_{\geq 0}$;
(2) The set of all partitions $\mathcal{P}$. The order of $\mathcal{P}$ can be defined as follows:
   $\forall \lambda,\, \mu \in \mathcal{P}$, $\lambda \geq \mu$ iff $|\lambda| > |\mu|$, or $|\lambda| = |\mu|$ and there exists a j such that $\lambda_i = \mu_j$ for $i \leq j-1$ and $\lambda_j > \mu_i$. The summation is taken to be " $\cup$ " and the zero-element is (0).
(3) $\mathcal{P}^n$. The order of $\mathcal{P}^n$ is defined similarly as (2):
   $\forall \overrightarrow{A},\, \overrightarrow{B} \in \mathcal{P}^n$, $\overrightarrow{A} \geq \overrightarrow{B}$ iff $\sum_{i=1}^{n} |A^i| > \sum_{i=1}^{n} |B^i|$, or $\sum_{i=1}^{n} |A^i| = \sum_{i=1}^{n} |B^i|$ and there is a j such that $A^i = B^i$ for $i \leq j-1$ and $A^j > B^j$.

Define

$$\overrightarrow{A} \cup \overrightarrow{B} = (A^1 \cup B^1, A^2 \cup B^2, ..., A^n \cup B^n)$$

$((0), (0), ..., (0))$ is the zero-element. Then $\mathcal{P}^n$ is a partitionable set.

Let S be a partitionable set. One can define partition with respect to S in the similar manner as that of $\mathbb{Z}_{\geq 0}$: a finite sequence of non-increasing non-minimum elements in S. We will call it an *S-partition*, $(\mathbf{0})$ the zero S-partition. Denote by $\mathcal{P}(S)$ the set of all S-partitions.

For an S-partition Λ, we can define the automorphism group of Λ in a similar way as that in the definition of traditional partition. Given $\beta \in S$, denote by $m_\beta(\Lambda)$ the number of times that β occurs in the parts of Λ, we then have

$$Aut\Lambda = \prod_{\beta \in S} m_\beta(\Lambda)!\,.$$

Introduce the following quantities associated with Λ,

$$\mathfrak{u}_\Lambda = \frac{\ell(\Lambda)!}{|Aut\Lambda|}\,, \qquad \Theta_\Lambda = \frac{(-1)^{\ell(\Lambda)-1}}{\ell(\Lambda)}\mathfrak{u}_\Lambda\,.$$

The following Lemma will be used in Section 4 to deduce the reformulated invariants.

Lemma 2.1 ([32], Lemma 2.3)**.** *Let S be a partionable set. If $f(t) = \sum_{n \geq 0} a_n t^n$, then*

$$f\left(\sum_{\substack{\beta \neq \mathbf{0} \\ \beta \in S}} A_\beta p_\beta(x)\right) = \sum_{\Lambda \in \mathcal{P}(S)} a_{\ell(\Lambda)} A_\Lambda p_\Lambda(x) \mathfrak{u}_\Lambda,$$

where

$$p_\Lambda(x) = \prod_{j=1}^{\ell(\Lambda)} p_{\Lambda_j}\,,\; A_\Lambda = \prod_{j=1}^{\ell(\Lambda)} A_{\Lambda_j}\,.$$

Proof. Note that

$$\left(\sum_{\substack{\beta \neq \mathbf{0} \\ \beta \in S}} \eta_\beta\right)^n = \sum_{\substack{\Lambda \in \mathcal{P}(S) \\ \ell(\Lambda) = n}} \eta_\Lambda \mathfrak{u}_\Lambda.$$

□

2.3. **Birman-Murakami-Wenzl Algebra.** The centralizer algebra

$$\operatorname{End}_{U_q(\mathfrak{so}(2N+1))}(V^{\otimes n}) = \{f \in \operatorname{End}(V^{\otimes n}) | fx = xf, \forall x \in U_q(\mathfrak{so}(2N+1))\} \tag{2.1}$$

for the standard representation of $U_q(\mathfrak{so}(2N+1))$ on $V = \mathbb{C}^{2N+1}$ is isomorphic, when $N > n$, to the Birman-Murakami-Wenzl algebra C_n.

Let $\mathbb{C}(t,q)$ be the field of rational functions with two variables. For each positive integer n, the Birman-Murakami-Wenzl algebra is defined to be an algebra C_n over $\mathbb{C}(t,q)$ as follows. The algebra C_1 is of one dimensional, and thus is identified to $\mathbb{C}(t,q)$. For $n > 1$, C_n is generated over $\mathbb{C}(t,q)$ by the generators $g_1, g_2, \cdots, g_{n-1}, e_1, e_2, \cdots, e_{n-1}$ and the relations

(A1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $1 \le i \le n-2$

(A2) $g_i g_j = g_j g_i$ if $|i-j| \geqslant 2$

(A3) $e_i g_i = t^{-1} e_i$

(A4) $e_i g_{i-1}^{\pm 1} e_i = t^{\pm 1} e_i$

(A5) $(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$.

The first two properties are the braiding relations. The following two properties are immediate from the above definition

(P1) $e_i^2 = x e_i$ for $x = 1 + \frac{t-t^{-1}}{q-q^{-1}}$

(P2) $(g_i - t^{-1})(g_i + q^{-1})(g_i - q) = 0$.

When the variable q, t approaches to 1, while $x = 1 + \frac{t-t^{-1}}{q-q^{-1}}$ is fixing, the above BMW algebra specializes to the Brauer algebra Br_n, which is semisimple and isomorphic to the centralizer algebra $\operatorname{End}_{\mathfrak{so}(2N+1)}(V^{\otimes n})$ if $N > n$, cf. [8] and also [53]. The Birman-Murakami-Wenzl algebras are semisimple except possibly when q takes the value of roots of unity or $t = q^m$ for some integer m. Obviousely, the BMW algebra is the deformation of the Brauer algebra.

2.4. **Irreducible Representations of Brauer Algebras.** For our purpose, we focus the generic case when the BMW Algebras C_n are semisimple. In this situation, its irreducible representation can be described similar to the Brauer Algebras Br_n.

As the centralizer algebra $\operatorname{End}_{\mathfrak{so}(2N+1)} V^{\otimes n}$, Br_n contains the group algebra $\mathbb{C}[S_n]$ as a direct summand, thus all the irreducible representations of S_n are also irreducible representations of Br_n, labeled by partitions of the integer n. Indeed, the set of irreducible representations of Br_n are bijective to the set of partitions of the integers $n - 2k$, where $k = 0, 1, \cdots, [\frac{n}{2}]$ [41, 51]. Thus the semi-simple algebra Br_n can be decomposed into the direct sum of simple algebras

$$Br_n \cong \bigoplus_{k=0}^{[\frac{n}{2}]} \bigoplus_{\lambda \vdash n-2k} M_{d_\lambda \times d_\lambda}(\mathbb{C}).$$

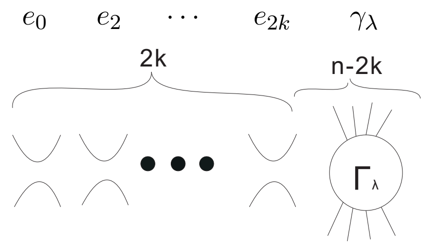
The work of Beliakova and Blanchet [4] constructed an explicit basis of the above decomposition. An up and down tableau $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a tube of n Young diagrams such that $\lambda_1 = (1)$ and each λ_i is obtained by adding or removing one box from λ_{i-1}. Let λ be a partition of $n - 2k$. Denote by $|\Lambda| = \lambda$ if $\lambda_n = \lambda$, and we say an up and down tableau Λ is of shape λ. There is a minimal path idempotent $p_\Lambda \in Br_n$ associated to each Λ. Then the minimal central idempotent π_λ of Br_n correspond to the irreducible representation labeled by λ is given by

$$\pi_\lambda = \sum_{|\Lambda| = \lambda} p_\Lambda.$$

In particular, the dimension of the irreducible representations d_λ is the number of up and down tableau of shape λ. More detail can be found in [4, 51].

The characters table and the orthogonal relations can be found in [40, 41, 42]. The values of a character of Br_n is completely determined by its values on the set of elements $e^k \otimes \gamma_\lambda$, where

e is the conjugacy class of $e_1, \cdots, e_{n-1}$ and γ_λ is the conjugacy class in S_{n-2k} labeled by the partition λ of $n-2k$. The notion $e^k \otimes \gamma_\lambda$ stands for the tangle in the following diagram.



where Γ_λ is a diagram in the conjugacy class of S_{n-2k} labeled by a partition λ of $n-2k$.

Denote χ_A the character of the irreducible representation of Br_n labeled by a partition $A \vdash n-2k$ for some k, and denote by $\chi_B^{S_n}$ the character of the irreducible representation of S_n labeled by a partition $B \vdash n$. It is known that when A is a partition of n, then $\chi_A(e^m \otimes \gamma_\lambda) = 0$ for all $m > 0$ and partition $\lambda \vdash n-2m$, and $\chi_A(\gamma_\mu) = \chi_A^{S_n}(\gamma_\mu)$ for partition $\mu \vdash n$ coincide with the characters of the permutation group S_n [41].

2.5. **Schur-Weyl Duality.** Both $\mathfrak{so}(2N+1)$ and Br_n acts on the tensor product $V^{\otimes n}$ and their actions commute each other. As a bi-module, $V^{\otimes n}$ has the following decomposition

$$V^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes U_\lambda,$$

where λ runs through all the partitions of $n, n-2, n-4, \cdots, 0$, V_λ (resp. U_λ) is the irreducible representation of $\mathfrak{so}(2N+1)$ (resp. Br_n) labeled by λ. A similar decomposition holds for the pair $U_q(\mathfrak{so}(2N+1))$ and C_n.

A power symmetric function of a sequence of variables $z = (z_i)_{i\in\mathbb{Z}}$ is defined by

$$pb_n(z) = (z_0)^n + \sum_{i=1}^{+\infty}[(z_i)^n + (z_{-i})^n].$$

For a partition λ,

$$pb_\lambda(z) = \prod_{j=1}^{\ell(\lambda)} pb_{\lambda_j}(z).$$

Denote $\widehat{Br}_n$ the set of all the characters of Br_n. For each partition A, we use sb_A to denote the type-B Schur function associated to A with infinitely many variables $z_0, z_{\pm1}, z_{\pm2}, \cdots$, which are completely determined by the system of equations inductively

$$x^k pb_\lambda = \sum_{A\in\widehat{Br}_n} \chi_A(e^{\otimes k}\otimes\gamma_\lambda) sb_A. \tag{2.2}$$

The parameter x is the structure constant in the definition of the Brauer algebra Br_n. The type-B Schur functions is independent of this parameter x, as one can see from the character formula of Brauer algebra, given by A. Ram in [41] Theorem 5.1. If A is a partition of n, then sb_A is a symmetric polynomial of degree n (not necessarily homogeneous).

Throughout this paper, we fix the following notations for partition set $\mathcal{P}^L$, where L is the number of components of link $\mathcal{L}$.

For $\overrightarrow{\mu} = (\mu^1, \mu^2, \cdots, \mu^L) \in \mathcal{P}^L$, denote

$$|\overrightarrow{\mu}| = (|\mu^1|, |\mu^2|, \cdots, |\mu^L|) \in \mathbb{Z}^L \tag{2.3}$$

and define

$$||\overrightarrow{\mu}|| = \sum_{\alpha=1}^{L} |\mu^\alpha|. \tag{2.4}$$

Write

$$\ell(\overrightarrow{\mu}) = \sum_{\alpha=1}^{L} \ell(\mu^\alpha) \tag{2.5}$$

for the sum of the length of each partition.

We denote $pb_{\overrightarrow{\mu}}(\overrightarrow{z}) = \prod_{\alpha=1}^{L} pb_{\mu^\alpha}(z_\alpha)$, where $z_\alpha = (z_{\alpha,i})_{i\in\mathbb{Z}}$.

Let $\widehat{Br}_{|\overrightarrow{\mu}|}$ denotes the set $\widehat{Br}_{|\mu^1|} \times \cdots \times \widehat{Br}_{|\mu^L|}$, then $\chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\mu}}) = \prod_{\alpha=1}^{L} \chi_{A^\alpha}(\gamma_{\mu^\alpha})$ for the character χ_{A^α} of $Br_{|\mu^\alpha|}$ labeled by A^α, a partition of $|\mu^\alpha| - 2k^\alpha$, and the conjugacy class γ_{μ^α} of Br_{d_α} labeled by μ^α.

3. Colored Kauffman Polynomials and Cabling Formula

3.1. **Colored Kauffman Polynomials (Orthogonal Quantum Groups Invariants).** Let B_m be the braid group of m strands which is generated by $\sigma_1, \cdots, \sigma_{m-1}$ with following defining relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & if\ |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & if\ |i-j| = 1 \end{cases} \tag{3.1}$$

Every link can be represented by the closure of some element in braid group B_m. This kind of braid representation is not unique. We fix such a braid representation, then we define the quantum group invariants of link via this braid. Finally we will see such kind of definition is independent of the choice of the braid representation.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and $U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra.

The ribbon category structure associated with $U_q(\mathfrak{g})$ is given by the following data:

(1) Associated to each pair of $U_q(\mathfrak{g})$-modules V and W, there is an isomorphism

$$\check{\mathcal{R}}_{V,W} : V \otimes W \to W \otimes V$$

such that

$$\begin{aligned} \check{\mathcal{R}}_{U\otimes V,W} &= (\check{\mathcal{R}}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{\mathcal{R}}_{V,W}) \\ \check{\mathcal{R}}_{U,V\otimes W} &= (\mathrm{id}_V \otimes \check{\mathcal{R}}_{U,W})(\check{\mathcal{R}}_{U,V} \otimes \mathrm{id}_W) \end{aligned}$$

for $U_q(\mathfrak{g})$-modules U, V, W.

Given $f \in \mathrm{Hom}_{U_q(\mathfrak{g})}(U, \widetilde{U})$, $g \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, \widetilde{V})$, one has the following naturality condition:

$$(g \otimes f) \circ \check{\mathcal{R}}_{U,V} = \check{\mathcal{R}}_{\widetilde{U},\widetilde{V}} \circ (f \otimes g).$$

(2) There exists an element $K_{2\rho} \in U_q(\mathfrak{g})$, called the enhancement of $\check{\mathcal{R}}$, such that

$$K_{2\rho}(v \otimes w) = K_{2\rho}(v) \otimes K_{2\rho}(w)$$

for any $v \in V$, $w \in W$. Here ρ is the half-sum of all positive roots of $\mathfrak{g}$.

Moreover, for every $z \in \mathrm{End}_{U_q(\mathfrak{g})}(V, W)$ with $z = \sum_i x_i \otimes y_i$, $x_i \in End(V)$, $y_i \in End(W)$ one has the quantum trace

$$\mathrm{tr}_W(z) = \sum_i \mathrm{tr}(y_i K_{2\rho}) \cdot x_i \in \mathrm{End}_{U_q(\mathfrak{g})}(V)$$

(3) For any $U_q(\mathfrak{g})$-module V, the ribbon structure $\theta_V : V \to V$ associated to V satisfies

$$\theta_V^{\pm 1} = \mathrm{tr}_V \check{\mathcal{R}}_{V,V}^{\pm 1}.$$

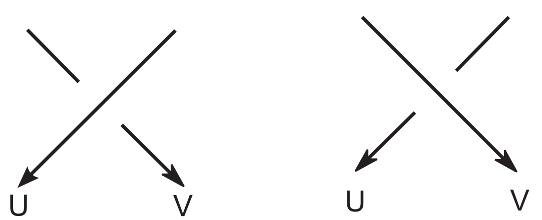
The ribbon structure also satisfies the following naturality condition

$$x \cdot \theta_V = \theta_{\widetilde{V}} \cdot x.$$

for any $x \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, \widetilde{V})$.

Let $\mathcal{L}$ be a link with L components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$, represented by the closure of $\beta \in B_m$. We associate each $\mathcal{K}_\alpha$ an irreducible representation V_{A^α} of quantized universal enveloping algebra $U_q(\mathfrak{g})$ labeled by highest weight A^α. In the sense of [41], these irreducible representations can be labeled by partitions. By abuse of notations, we use A^α's to denote those partitions. Let $i_1, \cdots, i_m$ be integers such that $i_k = \alpha$ if the k-th strand of β belongs to the α-th component of $\mathcal{L}$.

Let U, V be two $U_q(\mathfrak{g})$-modules labeling two outgoing strands of the crossing, the braiding $\check{\mathcal{R}}_{U,V}$ (resp. $\check{\mathcal{R}}_{V,U}^{-1}$) is assigned as in following figure.



$\check{\mathcal{R}}_{U,V}$ $\check{\mathcal{R}}_{V,U}^{-1}$

Assign crossing by $\check{\mathcal{R}}$.

The above assignment will give a representation of B_m on $U_q(\mathfrak{g})$-module $V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}$. Namely, for any generator $\sigma_j \in B_m$,

define

$$h(\sigma_j) = \mathrm{id}_{V_{A^{i_1}}} \otimes \cdots \otimes \check{\mathcal{R}}_{V_{A^{i_{j+1}}}, V_{A^{i_j}}} \otimes \cdots \otimes \mathrm{id}_{V_{A^{i_m}}},$$

and

$$h(\sigma_j^{-1}) = \mathrm{id}_{V_{A^{i_1}}} \otimes \cdots \otimes \check{\mathcal{R}}^{-1}_{V_{A^{i_j}}, V_{A^{i_{j+1}}}} \otimes \cdots \otimes \mathrm{id}_{V_{A^{i_m}}},$$

Therefore, any link $\mathcal{L}$ will provide an isomorphism

$$h(\beta) \in \mathrm{End}_{U_q(\mathfrak{g})}(V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}).$$

The representation of braid group B_n on $V^{\otimes n}$ factors through the BMW algebra C_n by sending σ_j to $g_j \in C_n$. By abuse of notations, we still denote this via $g_j = h(\sigma_j)$.

The quantum trace

$$\mathrm{tr}_{V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}} h(\beta)$$

defines framing dependent link invariant of link $\mathcal{L}$.

In order to eliminate the framing dependency, we make the following refinement [29]

$$W^{\mathfrak{so}(2N+1)}_{V_{A^1},\cdots,V_{A^L}}(\mathcal{L};q)=\theta_{V_{A^1}}^{-w(\mathcal{K}_1)}\cdots\theta_{V_{A^L}}^{-w(\mathcal{K}_L)}\mathrm{tr}_{V_{A^{i_1}}\otimes\cdots\otimes V_{A^{i_m}}}(h(\beta)),$$

where $w(\mathcal{K}_\alpha)$ is the writhe number of $\mathcal{K}_\alpha$ in β, i.e., the number of positive crossing minus the number of negative crossings.

The above quantity is invariant under the Markov moves, hence is an invariant of the underlying link $\mathcal{L}$.

Quantum group invariants of links can be defined over any complex simple Lie algebra $\mathfrak{g}$. However, in this paper, we mainly consider the quantum group invariants of links defined over $\mathfrak{so}(2N+1)$. More generally, one can also include the case for $\mathfrak{so}(2N)$ and $\mathfrak{sp}(2N)$; however, we will not do so, since the quantum group invariants associated to these Lie algebras all give the colored Kauffman polynomials. To distinguish $U_q(\mathfrak{so}(2N+1))$ from the quantum group corresponding to spin group, we only consider those representations parameterized by the highest weights in the root lattice of the Lie group $SO(2N+1)$, instead of the spin group. These highest weights are, similar to the case of $\mathfrak{sl}_N$, partitions of length at most N, i.e $\{\mu|\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_N\geqslant 0\}$.

Let's consider $U_q(\mathfrak{so}(2N+1))$, the quantized universal enveloping algebra of orthogonal lie algebra $\mathfrak{so}(2N+1)$, which is generated by $\{H_i,X_i^+,X_i^-\}$ together with the following defining relations:

$$[H_i,H_j]=0,\ [H_i,X_j^{\pm}]=\pm(C)_{ij}X_j^{\pm}\text{ and }[X_i^+,X_j^-]=\delta_{ij}\frac{q^{H_i}-q^{-H_i}}{q-q^{-1}},$$

where C is the Cartan matrix of $\mathfrak{g}=\mathfrak{so}(2N+1)$

and the Serre type relations

$$\sum_{k=0}^{1-(C)_{ij}}(-1)^k\frac{[1-(C)_{ij}]_q!}{[1-(C)_{ij}-k]_q![k]_q!}(X_i^{\pm})^{1-(C)_{ij}-k}X_j^{\pm}(X_i^{\pm})^k=0,\text{ for all }i\neq j,$$

where $[k]_q!=\prod_{i=1}^{k}[i]_q$ and the q-number is defined as

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}.$$

When $q\to 1$, the universal enveloping algebra $U_q(\mathfrak{so}(2N+1))$ reduces to the Lie algebra $\mathfrak{so}(2N+1)$.

Drinfeld [12] defined the universal $\mathcal{R}$-matrix of $U_q(\mathfrak{g})$ as

$$\mathcal{R}=q^{\sum\limits_{i,j}(C^{-1})_{ij}H_i\otimes H_j}\prod_{\beta\in\triangle^+}\exp_q[(1-q^{-2})X_\beta^+\otimes X_\beta^-],\tag{3.2}$$

where $\triangle^+$denotes the set of positive roots and the q-exponential is of the form

$$\exp_q(x)=\sum_{k=0}^{\infty}q^{\frac{1}{2}k(k+1)\frac{x^k}{[k]_q!}}.$$

The ribbon category structure is defined by letting $\check{R}=P_{12}\mathcal{R}$ for the above universal $\mathcal{R}$-matrix, and taking $K_{2\rho}$ to be $q^{-\rho^*}$. The operator $P_{12}:V\otimes W\to W\otimes V$ switches the two components, and ρ^* denotes the element in the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ corresponding to ρ.

The positive roots of $\mathfrak{so}(2N+1)$ are given by $\vartheta_i\pm\vartheta_j$ for $1\leqslant i<j\leqslant N$ and $\vartheta_1,\vartheta_2,\cdots,\vartheta_N$, where ϑ_i has eigenvalue x_i when acting on the matrix element

$$\mathrm{diag}\{-x_N,-x_{N-1},\cdots,-x_1,0,x_1,\cdots,x_{N-1},x_N\}$$

in the Cartan subalgebra. The sum of the positive roots is given by

$$2\rho=\sum_{i=1}^{N}\vartheta_i+\sum_{1\leqslant i<j\leqslant N}[(\vartheta_i-\vartheta_j)+(\vartheta_i+\vartheta_j)]=\sum_{i=1}^{N}(2N+1-2i)\vartheta_i,$$

and

$$K_{2\rho}=\mathrm{diag}\{q^{1-2N},q^{3-2N},\cdots,q^{-3},q^{-1},1,q,q^{3}\cdots,q^{2N-3},q^{2N-1}\}.$$

Alternatively, we can write

$$K_{2\rho}(v_i)=\begin{cases} q^{2i-1-2N}v_i & 1\leq i\leq N\\ v_i & i=N+1\\ q^{2i-3-2N}v_i & N+2\leq i\leq 2N+1\end{cases}.$$

The universal matrix $\check{\mathcal{R}}$ acting on $V\otimes V$ for the natural representation of $U_q(\mathfrak{so}(2N+1))$ on V is given by Turaev [50]:

$$\begin{aligned}\check{\mathcal{R}}=&q\sum_{i\neq N+1}E_{i,i}\otimes E_{i,i}+E_{N+1,N+1}\otimes E_{N+1,N+1}+\sum_{j}\sum_{\substack{i\neq j\\ i\neq 2N+2-j}}E_{j,i}\otimes E_{i,j}\\ &+q^{-1}\sum_{i\neq N+1}E_{2N+2-i,i}\otimes E_{i,2N+2-i}+(q-q^{-1})\sum_{i<j}E_{i,i}\otimes E_{j,j}\\ &-(q-q^{-1})\sum_{i<j}q^{\bar{i}-\bar{j}}E_{2N+2-j,i}\otimes E_{j,2N+2-i},\end{aligned}$$

where $E_{i,j}$ is the $(2N+1)\times(2N+1)$ matrix with

$$(E_{i,j})_{kl}=\begin{cases}1 & (k,l)=(i,j)\\ 0 & elsewhere\end{cases}$$

and

$$\bar{i}=\begin{cases} i+\frac{1}{2} & 1\leq i\leq N\\ i & i=N+1\\ i-\frac{1}{2} & N+2\leq i\leq 2N+1\end{cases}.$$

The ribbon structure $\theta_{V_{A^\alpha}}$ is equal to $q^{<A^\alpha,A^\alpha+2\rho>}$ for $1\leq\alpha\leq L$.

Define the orthogonal quantum group invariants $W^{SO}_{A^1,\cdots,A^L}(\mathcal{L},q,t)\in\mathbb{C}(q,t)$ such that

$$W^{SO}_{A^1,\cdots,A^L}(\mathcal{L};q,q^{2N})=W^{\mathfrak{so}(2N+1)}_{V_{A^1},\cdots,V_{A^L}}(\mathcal{L};q)=q^{\sum_{\alpha=1}^{L}-<A^\alpha,A^\alpha+2\rho>w(\mathcal{K}_\alpha)}\mathrm{tr}_{V_{A^{i_1}}\otimes\cdots\otimes V_{A^{i_m}}}(h(\beta)).\tag{3.3}$$

Then we want to compute the identity $q^{<A^\alpha,A^\alpha+2\rho>}$. We will first introduce the representation theory of the BMW algebra.

From now on, we only restrict ourselves in the case when the Birman-Murakami-Wenzl algebra C_n is semisimple and N is large. The representations of C_n can be described in the same way as the Brauer algebra Br_n. The semi-simplicity implies that the representation $V^{\otimes n}$ of C_n admits a direct sum decomposition

$$V^{\otimes n}=\bigoplus_{\lambda\in\widehat{Br}_n}d_\lambda\cdot V_\lambda.$$

The multiplicities d_λ are all positive integers. In particular, any irreducible representation V_A of $U_q(so(2N+1))$ appear as a direct summand of $V^{\otimes r}$ for integer $r=|A|,|A|+2,|A|+4,\cdots$.

By Schur lemma,

$$C_n \cong \mathrm{End}_{U_q(\mathfrak{so}(2N+1))} V^{\otimes n} \cong \bigoplus_{\lambda \in \widehat{Br}_n} C_\lambda,$$

where $C_\lambda = \mathrm{End}_{U_q(\mathfrak{so}(2N+1))}(d_\lambda V_\lambda)$ is isomorphic to the $d_\lambda \times d_\lambda$ matrix algebra, labelled by the characters $\widehat{Br}_n$ of Br_n as the decomposition of $V^{\otimes n}$.

A *minimal idempotent* $p \in C_n$ satisfies $p^2 = p$ and the action of $U_q(\mathfrak{so}(2N+1))$ on the subspace $p \cdot V^{\otimes n}$ is an irreducible representation. Another description of p is that, there exist exactly one $\lambda \in \widehat{Br}_n$ such that the restriction of p to C_λ is non-zero, and it's a diagonalizable matrix with exactly one eigenvalue 1 and all others 0.

Let y be an element in C_n, and the normal (or say, non-quantum) trace of its λ component via the above isomorphism is denoted by $\zeta_n^\lambda(y)$. Since y and all the idempotents are elements in C_n, they are finite linear combinations of products of the generators g_i's and e_i's, which imply $\zeta_n^\lambda(y)$ is, in general, a rational function of q and t.

It is not hard to get the following identity from the Turaev's [50] construction of universal matrix $\check{\mathcal{R}}$ (See Section 10 for detail):

$$\theta_V = q^{2N} \cdot \mathrm{id}_V,$$

where V is the standard representation of $U_q(\mathfrak{so}(2N+1))$ on $\mathbb{C}^{2N+1}$.

More generally, we have the following lemma obtained by Reshetikhin [43]

Lemma 3.1. *For each partition $\lambda \vdash n - 2f$ with $\ell(\lambda) \leq N$, one has*

$$\theta_{V_\lambda} = q^{\kappa_\lambda + 2N(n-2f)} \cdot \mathrm{id}_{V\lambda},$$

where $\kappa_\lambda = \prod_{j=1}^{\ell(\lambda)} \lambda_j(\lambda_j - 2j + 1)$.

This result can be understand in the following way. First we have

$$\theta_V = q^{2N} \cdot \mathrm{id}_V.$$

A result of Aiston-Morton (Theorem 5.5 of [2], cf., Theorem 4.1 of [29]) states that

$$\theta_{V_\lambda} = q^{\kappa_\lambda + nN - \frac{n^2}{N}} \cdot \mathrm{id}_{V\lambda}.$$

In [29], they use a different normalization for universal $\check{\mathcal{R}}$-matrices, thus they have

$$q^{\frac{1}{N}} \theta_V = q^N \cdot \mathrm{id}_V$$

and also a different corresponding normalization for $h : \mathbb{C}B_n \to C_n(V)$ factoring through the Hecke algebra $\mathcal{H}_n(q)$ via

$$q^{\frac{1}{N}} \sigma_i \mapsto g_i \mapsto q^{\frac{1}{N}} h_V(\sigma_i)$$

Then we translate their normalization to ours, i.e.,

$$\sigma_i \mapsto g_i \mapsto h(\sigma_i),$$

$$\theta_V = q^N \cdot \mathrm{id}_V$$

and

$$\theta_{V_\lambda} = q^{\kappa_\lambda + nN} \cdot \mathrm{id}_{V\lambda}.$$

Then it is quite easy to get

$$\theta_{V_\lambda} = q^{\kappa_\lambda + 2N(n-2f)} \cdot \mathrm{id}_{V\lambda},$$

Now we can write down the explicit formula for orthogonal quantum group invariants as follows

$$W^{SO}_{A^1,\cdots,A^L}(\mathcal{L};q,t)=q^{-\sum\limits_{\alpha=1}^{L}\kappa_{A^\alpha}w(\mathcal{K}_\alpha)}t^{-\sum\limits_{\alpha=1}^{L}|A^\alpha|w(\mathcal{K}_\alpha)}\cdot \mathrm{tr}_{V_{A^{i_1}}\otimes\cdots\otimes V_{A^{i_m}}}(h(\beta)) \tag{3.4}$$

for all sufficiently integers N. In particular, when the link is trivial with L components, the quantum group invariant is the product of quantum dimensions

$$W^{SO}_{A^1,\cdots,A^L}(\bigcirc^L;q,q^{2N})=\prod_{\alpha=1}^{L}\dim_q(V_{A^\alpha}). \tag{3.5}$$

The quantum dimension is computed by Wenzl [52], which we quote it here. Let λ be a partition. We also identify it to the corresponding Young diagram. For each pair of positive integers (i,j), define

$$h(i,j)=\lambda_i+\lambda'_j-i-j+1$$

to be the *hook length*, where λ' is the transposed Young diagram of λ. Also define

$$d(i,j)=\begin{cases}\lambda_i+\lambda_j-i-j+1 & i\le j\\ -\lambda'_i-\lambda'_j+i+j-1 & i>j\end{cases}$$

Theorem 3.2 (Wenzl [52]). *Let λ be a Young diagram with m rows and let $\mathcal{Q}_\lambda(t,q)$ be the rational function given by*

$$\mathcal{Q}_\lambda(t,q)=\prod_{(j,j)\in\lambda}\left(1+\frac{tq^{\lambda_j-\lambda'_j}-t^{-1}q^{\lambda'_j-\lambda_j}}{[h(j,j)]_q}\right)\prod_{\substack{(i,j)\in\lambda\\ i\neq j}}\frac{tq^{d(i,j)}-t^{-1}q^{-d(i,j)}}{[h(i,j)]_q} \tag{3.6}$$

Then the quantum trace $\dim_q V_\lambda$ of the representation of $U_q(\mathfrak{so}(2N+1))$ corresponding to λ equal to $\mathcal{Q}_\lambda(q^{2N},q)$ for all $N>|\lambda|$.

In the above expression, if we fix t and let q tends to 1, the pole order of $\mathcal{Q}_\lambda(t,q)$ is $|\lambda|$, the number of boxes in the Young diagram. The poles order at $q=1$ of the quantum group invariant of unknots in (3.5) is $||\overrightarrow{A}||=\sum\limits_{\alpha=1}^{L}|A^\alpha|$.

The special value $sb_A(q^{1-2N},q^{3-2N},\cdots,q^{-1},1,q,...,q^{2N-3},q^{2N-1})=\mathcal{Q}_\lambda(q,q^{2N})$ is the quantum dimension $\dim_q(V_A)$, denoted also by $sb_A(q,t)$. Here we only evaluate the function in the variables

$$z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N,$$

and setting all the rest variables equal to zero.

The quantum dimension of small partitions can be found in Section 10, where we use the symbol of type-B Schur function $sb_A(q,t)$.

Similar to type-A Schur function, type-B Schur function has the following expansion

$$sb_\lambda(z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N)=\sum_{\substack{\mu\models n\\ \ell(\mu)\le 2N+1}}\dim(p_\lambda V^{\otimes n}\cap M^\mu)\cdot\prod_{i=-N}^{N}z_i^{\mu_{(i+N+1)}},$$

where M^μ is called the permutation module defined by

$$M^\mu=\left\{v\in V^{\otimes n}|H_i(v)=q^{\mu_i-\mu_{i+1}}v \text{ for } i=1,2,...,N-1 \text{ and } H_N(v)=q^{\mu_N}v\right\},$$

and $\dim(p_\lambda V^{\otimes n}\cap M^\mu)$ is called the *Kostka number.*

When all the representations $A^1, \cdots, A^L$ are the natural representation of $\mathfrak{so}(2N+1)$ on $\mathbb{C}^{2N+1}$, i.e., the partitions A^α all equal to (1), the invariant

$$W^{SO}_{A^1,\cdots,A^L}(\mathcal{L},q,t) = t^{2lk(\mathcal{L})}(1+\frac{t-t^{-1}}{q-q^{-1}})K_{\mathcal{L}}(q,t)$$

for the Kauffman polynomial $K_{\mathcal{L}}(q,t)$, where we normalized the Kauffman polynomials such that $K_{\bigcirc}(q,t)=1$. The orthogonal group invariants $W^{SO}_{A^1,\cdots,A^L}(\mathcal{L};q,t)$ for general A^α are also called colored Kauffman polynomials.

It is normally very hard to calculate these quantum group invariants. Anyway, we can simplify the computation a lot with the help of cabling technique discussed in the next subsection.

3.2. **Cabling technique.** The following Lemma proved by Xiao-Song Lin and Hao Zheng [29] reduce the study of quantum group invariants of arbitrary representations to the study of the links and minimal idempotents.

Lemma 3.3 ([29], Lemma 3.3)**.** *Let $\beta \in B_m$ be a braid and $p_\alpha \in C_{d_\alpha}, \alpha = 1,\cdots,L$ be L minimal idempotents corresponding to the irreducible representations $V_{A^1},\cdots,V_{A^L}$, where A^α denote the partition of $|A^\alpha| = d_\alpha$ labelling V_{A^α}. Denote $\overrightarrow{d} = (d_1,\cdots,d_L)$ and let $i_1,\cdots,i_m$ be integers such that $i_k = \alpha$ if the k-th strand of β belongs to the α-th component of $\mathcal{L}$. Let $\beta_{\overrightarrow{d}}$ be the cabling braid of β, replacing the k-th strand of β by d_{i_k} parallel ones. Then*

$$\mathrm{tr}_{V_{A^{i_1}}\otimes\cdots\otimes V_{A^{i_m}}} h(\beta) = \mathrm{tr}_{V^{\otimes n}}[h(\beta_{\overrightarrow{d}})\cdot(p_{i_1}\otimes\cdots\otimes p_{i_m})], \tag{3.7}$$

where $n = d_{i_1}+d_{i_2}+\cdots+d_{i_m}$.

One immediately gets the following lemmas proved in [29] and reformulated into the setting of orthogonal group case.

Lemma 3.4. *For any element $y \in C_n$,*

$$\mathrm{tr}_{V^{\otimes n}} y = \sum_{k=0}^{[\frac{n}{2}]}\sum_{\lambda\vdash n-2k}\zeta_n^\lambda(y)\cdot sb_\lambda(q^{1-2N},q^{3-2N},\cdots,q^{-1},1,q,\cdots,q^{2N-3},q^{2N-1}). \tag{3.8}$$

For any braid $\beta\in B_m$, taking $y = h(\beta_{\overrightarrow{d}})\cdot(p_{i_1}\otimes p_{i_2}\otimes\cdots\otimes p_{i_m})$, where the closure of β is the link $\mathcal{L}$. The setting is same as that in 3.3 after replacing q^{2N} by t, we have

$$W^{SO}_{\overrightarrow{A}}(\mathcal{L},q,t) = q^{\sum_{\alpha=1}^{L}-\kappa_{A^\alpha}w(\mathcal{K}_\alpha)}t^{-\sum_{\alpha=1}^{L}|A^\alpha|w(\mathcal{K}_\alpha)}\cdot\sum_{k=0}^{[\frac{n}{2}]}\sum_{\lambda\vdash n-2k}\zeta_n^\lambda(h(\beta_{\overrightarrow{d}})\cdot(p_{i_1}\otimes p_{i_2}\otimes\cdots\otimes p_{i_m}))\cdot\mathcal{Q}_\lambda(q,t), \tag{3.9}$$

where $n = |A^{i_1}|+\cdots+|A^{i_m}|$.

3.3. **An Explicit Formula of Colored Kauffman Polynomials for Torus Links.** The coefficients $\zeta_n^\lambda(h(\beta_{\overrightarrow{d}})\cdot(p_{i_1}\otimes p_{i_2}\otimes\cdots\otimes p_{i_m}))$ in (3.9) are usually hard to compute. However, they are computable for torus links. The torus link $T(r,k)$ is the closure of $(\delta_r)^k = (\sigma_1\cdots\sigma_{r-1})^k$. It is a knot if and only if $(r,k)=1$. For example, $T(2,3)$ is the trefoil knot, and $T(2,2)$ is the Hopf link. We developed the following method in this subsection based on the work of Lin-Zheng [29].

Lemma 3.5. *For each partition $\lambda\vdash(n-2f)$ where $f = 0,1,\cdots,[\frac{n}{2}]$, we have*

$$h((\delta_n)^n)\cdot p_\lambda = q^{\kappa_\lambda-4fN}\cdot p_\lambda = q^{\kappa_\lambda}t^{-2f}\cdot p_\lambda. \tag{3.10}$$

Proof. Again write V for the standard representation of $U_q(\mathfrak{so}(2N+1))$ on the vector space $\mathbb{C}^{2N+1}$.

From Lemma 3.1, for each partition $\lambda \vdash n-2f$ with $\ell(\lambda)\leq N$, one has

$$\theta_{V_\lambda}=q^{\kappa_\lambda+2N(n-2f)}\cdot \mathrm{id}_{V_\lambda},$$

Substitute the above formula to the following formula proved in Lemma 3.2 of [29]

$$(\theta_V^{\otimes n}\cdot h((\delta_n)^n)\cdot p_\lambda=\theta_{V_\lambda}\cdot p_\lambda$$

and the result follows. □

In the following, we assume $z_0=1$ and $z_{-n}z_n=1$ for all positive integer $n=1,2,\cdots,N$, i.e, the matrix $\mathrm{diag}(z_{-N},z_{1-N},\cdots z_{-1},z_0,z_1,\cdots,z_{N-1},z_N)$ is a generic element in the maximal torus of $SO(2N+1,\mathbb{C})$. Let the constants $\widetilde{c}^{\lambda}_{\overrightarrow{A}}$ be the rational number determined by equations (3.11)

$$\prod_{\alpha=1}^{L} sb_{A^\alpha}(z_{-N}^r,\cdots,z_{-1}^r,z_0^r,z_1^r,\cdots,z_N^r)=\sum_{f=0}^{[rn/2]}\sum_{\lambda\vdash rn-2f}\widetilde{c}^{\lambda}_{\overrightarrow{A}}\cdot sb_\lambda(z_{-N},\cdots,z_{-1},z_0,z_1,\cdots,z_N).$$

Theorem 3.6. *Let $\mathcal{L}$ be the torus link $T(rL,kL)$ with r,k relatively prime. A^α is a partition of d_α for each $\alpha=1,2,\cdots,L$ and $n=d_1+d_2+\cdots+d_L$. Then*

$$W^{SO}_{\overrightarrow{A}}(\mathcal{L},q,t)=q^{-kr\sum_{\alpha=1}^{L}\kappa_{A^\alpha}}\cdot t^{-k(r-1)n}\cdot\sum_{f=0}^{[\frac{nr}{2}]}\sum_{\lambda\vdash(rn-2f)}\widetilde{c}^{\lambda}_{\overrightarrow{A}}\cdot q^{\frac{k\kappa_\lambda}{r}}t^{-\frac{2fk}{r}}\cdot sb_\lambda(q,t)\tag{3.12}$$

Theorem 3.6 gives an explicit formula of the orthogonal quantum group invariants (colored Kauffman polynomials) of torus links in terms of constants $\widetilde{c}^{\lambda}_{\overrightarrow{A}}$. In [48], Sebastien Stevan generalize this result to all classic gauge group and cable knots. In Section 5, we use this formula to verify certain cases of Conjecture 5.1. The proof of Theorem 3.6 follows from the Cabling formula (3.9), Lemma 3.5 and the following Lemma 3.7.

Lemma 3.7. *Let $n=\|\overrightarrow{A}\|$, where $A^\alpha\vdash d_\alpha$, and let r and k be two relatively prime positive integers. Take $\beta\in B_{rn}$ to be the braid obtained by cabling the $(iL+j)$-th strand of $(\delta_{rL})^{kL}$ to $|A^j|$ parallel ones. For each partition $\lambda\vdash(rn-2f)$, where $f=0,1,2,\cdots,[\frac{rn}{2}]$, we have*

$$\zeta^{\lambda}_{rn}(h(\beta)\cdot(p_{A^1}\otimes\cdots\otimes p_{A^L})^{\otimes r})=\widetilde{c}^{\lambda}_{\overrightarrow{A}}\cdot q^{-k\sum_{\alpha=1}^{L}\kappa_{A^\alpha}+\frac{k\kappa_\lambda}{r}}t^{-\frac{2kf}{r}}.\tag{3.13}$$

Proof. Write $p_{\overrightarrow{A}}=p_{A^1}\otimes\cdots\otimes p_{A^L}$ and let π_λ be the unit of C_λ. Obviously π_λ is in the center of C_{rn}. A slightly non-obvious fact is that $h(\beta)$ commutes with $p^{\otimes n}_{\overrightarrow{A}}$ following from the naturality of $\check{\mathcal{R}}$. Let

$$x_\lambda=\pi_\lambda\cdot h(\beta)\cdot p^{\otimes r}_{\overrightarrow{A}}\tag{3.14}$$

be a matrix in C_λ, whose trace is

$$\mathrm{tr}(x_\lambda)=\zeta^\lambda(h(\beta)\cdot p^{\otimes r}_{\overrightarrow{A}}).\tag{3.15}$$

The Cabling of torus link has the nice property

$$h(\beta^r)=h((\delta_{rn})^{krn})\cdot(h((\delta_{d_1})^{-kd_1})\otimes\cdots\otimes h((\delta_{d_L})^{-kd_L}))^{\otimes r}.\tag{3.16}$$

The previous Lemma 3.5 then imply

$$x_\lambda^r=\pi_\lambda\cdot h(\beta^r)\cdot p^{\otimes r}_{\overrightarrow{A}}=q^{-kr\sum_{\alpha=1}^{L}\kappa_{A^\alpha}+k\kappa_\lambda}t^{-2kf}\cdot\pi_\lambda\cdot p^{\otimes r}_{\overrightarrow{A}}.\tag{3.17}$$

Thus the eigenvalues of x_λ are either 0 or $q^{-kr\sum_{\alpha=1}^{L}\kappa_{A^\alpha}+\frac{k\kappa_\lambda}{r}}t^{\frac{-2kf}{r}}$ times a r-th root of unity. Together with the fact that $\mathrm{tr}(x_\lambda)\in\mathbb{Q}(q,t)$, we see that $\mathrm{tr}(x_\lambda)=a^\lambda\cdot q^{-k\sum_{\alpha=1}^{L}\kappa_{A^\alpha}+\frac{k\kappa_\lambda}{r}}t^{-\frac{2kf}{r}}$ for some $a^\lambda\in\mathbb{Q}$ independent of q and t.

The rest is to compute this rational number a^λ. Passing to the limit $q\to1$ and $t\to1$. The element $h(\beta)$ reduce to a permutation $\tau\in S_{rn}$ acting cyclicly on the $V^{\otimes n}$-factors of

$$V^{\otimes rn}=V^{\otimes n}\otimes\cdots\otimes V^{\otimes n}.$$

$$\begin{aligned}
&\sum_{f=0}^{[\frac{rn}{2}]}\sum_{\lambda\vdash(rn-2f)}a^\lambda\cdot sb_\lambda(z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N)\\
=&\sum_{f=0}^{[\frac{rn}{2}]}\sum_{\lambda\vdash(rn-2f)}a^\lambda\sum_{\substack{\mu\models rn\\ \ell(\mu)\leq 2N+1}}\dim(p_\lambda V^{\otimes rn}\cap M^\mu)\cdot\prod_{i=-N}^{N}z_i^{\mu(i+N+1)}\\
=&\sum_{\substack{\mu\models rn\\ \ell(\mu)\leq 2N+1}}\mathrm{tr}(\tau|_{p_{\overrightarrow{A}}^{\otimes r}V^{\otimes rn}\cap M^\mu})\cdot\prod_{i=-N}^{N}z_i^{\mu(i+N+1)}\\
=&\sum_{\substack{\mu\models n\\ \ell(\mu)\leq 2N+1}}\dim(p_{\overrightarrow{A}}V^{\otimes n}\cap M^\mu)\cdot\prod_{i=-N}^{N}z_i^{r\mu(i+N+1)}\\
=&\prod_{\alpha=1}^{L}[\sum_{\substack{\mu\models n_\alpha\\ \ell(\mu)\leq 2N+1}}\dim(p_{A^\alpha}V^{\otimes n_\alpha}\cap M^\mu)\cdot\prod_{i=-N}^{N}z_i^{r\mu(i+N+1)}]\\
=&\prod_{\alpha=1}^{L}sb_{A^\alpha}(z_{-N}^r,z_{1-N}^r,\cdots,z_{-1}^r,z_0^r,z_1^r,\cdots,z_{N-1}^r,z_N^r).
\end{aligned}$$

Compare to (3.11), $a^\lambda=\widetilde{c}^{\lambda}_{\overrightarrow{A}}$. □

Remark 3.1. Similar computations by starting with $U_q(\mathfrak{sp}(2N))$ and $U_q(\mathfrak{so}(2N))$ lead to the same theorem for Kauffman polynomials. Thus together with the type-A analog proved in [29] Theorem 5.1, these computations provide formulas of quantum group invariants of Torus links associated to simple Lie-algebras of type A, B, C and D.

4. Orthogonal Chern-Simons Partition Function

4.1. **Partition Function.** The *orthogonal Chern-Simons partition function* of $\mathcal{L}$ is defined by

$$Z_{CS}^{SO}(\mathcal{L};q,t)=\sum_{\overrightarrow{\mu}\in\mathcal{P}^L}\frac{pb_{\overrightarrow{\mu}}(\overrightarrow{z})}{\mathfrak{z}_{\overrightarrow{\mu}}}\cdot\sum_{\overrightarrow{A}\in\widehat{Br}_{|\overrightarrow{\mu}|}}\chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\mu}})W_{\overrightarrow{A}}^{SO}(\mathcal{L};q,t).\tag{4.1}$$

The above definition is motivated from physicists' path integral approach [6], and it is different from the definition given by Equation (4.10) in [6]. Unlike the $SU(N)$ Chern-Simons partition

function, the above $Z_{CS}^{SO}(\mathcal{L};q,t)$ can not be simplified to

$$Z_{CS}^{SO}(\mathcal{L};q,t) = \sum_{\overrightarrow{A}\in\mathcal{P}^L} W_{\overrightarrow{A}}^{SO}(\mathcal{L};q,t) sb_{\overrightarrow{A}}(\overrightarrow{z}). \tag{4.2}$$

Because orthogonality of type-A Schur function fails in type-B case [40, 41, 42].

Define the *free energy* as follows

$$F^{SO}(\mathcal{L};q,t) = \log Z_{CS}^{SO}(\mathcal{L};q,t). \tag{4.3}$$

The partition function of unknots with L components can be computed explicitly (See Proposition 10.2 for detail). In fact we have the following expression for the free energy

$$F^{SO}(\bigcirc^L;q,t) = \sum_{n=1}^{+\infty} \frac{1}{n}\cdot(1+\frac{t^n-t^{-n}}{q^n-q^{-n}})\cdot\sum_{\alpha=1}^{L} pb_n(z_\alpha), \tag{4.4}$$

4.2. **Reformulated Invariants.** The reformulated link invariants are rational functions $g_{\overrightarrow{A}}(q,t)\in\mathbb{C}(q,t)$ determined by the expansion

$$F^{SO}(\mathcal{L};q,t) = \sum_{d=1}^{\infty}\sum_{\overrightarrow{\mu}\neq\overrightarrow{0}} \frac{1}{d} g_{\overrightarrow{\mu}}(q^d,t^d)\prod_{\alpha=1}^{L} pb_{\mu^\alpha}((z_\alpha)^d). \tag{4.5}$$

As in [25], define the operator ψ_d by

$$\psi_d\circ F(q,t;pb(\overrightarrow{z})) = F(q^d,t^d;pb(\overrightarrow{z}^d)). \tag{4.6}$$

Then define the *plethystic exponential* [15]

$$\mathrm{Exp}(F) = \exp(\sum_{k=1}^{+\infty}\frac{\psi_k}{k}\circ F) \tag{4.7}$$

and its inverse

$$\mathrm{Log}(F) = \sum_{k=1}^{+\infty}\frac{\mu(k)}{k}\log(\psi_k\circ F), \tag{4.8}$$

where $\mu(k)$ is the Möbius function. In terms of these operators, we could write

$$Z_{CS}^{SO}(\mathcal{L};q,t) = \mathrm{Exp}(\sum_{\overrightarrow{\mu}\neq\overrightarrow{0}} g_{\overrightarrow{\mu}}(q,t)\prod_{\alpha=1}^{L} pb_{\mu^\alpha}(z_\alpha)). \tag{4.9}$$

If we expand the partition function

$$Z_{CS}^{SO}(\mathcal{L};q,t) = 1+\sum_{\overrightarrow{\mu}\neq\overrightarrow{0}} Z_{\overrightarrow{\mu}}^{SO} pb_{\overrightarrow{\mu}}(\overrightarrow{z}), \tag{4.10}$$

where $Z_{\overrightarrow{\mu}}^{SO}(\mathcal{L};q,t) = \sum_{\overrightarrow{A}\in\widehat{Br}_{|\overrightarrow{\mu}|}} \frac{\chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\mu}})}{\mathfrak{z}_{\overrightarrow{\mu}}} W_{\overrightarrow{A}}^{SO}(\mathcal{L};q,t)$.

If we expand the free energy

$$F^{SO}(\mathcal{L};q,t) = \sum_{\overrightarrow{\mu}\neq\overrightarrow{0}} F_{\overrightarrow{\mu}}^{SO} pb_{\overrightarrow{\mu}}(\overrightarrow{z}). \tag{4.11}$$

From Lemma 2.1 (Lemma 2.3 in [32]), we have

$$F^{SO}_{\overrightarrow{\mu}} = \sum_{\substack{\Lambda\in\mathcal{P}(\mathcal{P}^L)\\ |\Lambda|=\overrightarrow{\mu}}} \frac{(-1)^{\ell(\Lambda)-1}\ell(\Lambda)!}{\ell(\Lambda)|Aut\Lambda|} Z^{SO}_{\Lambda}. \tag{4.12}$$

Clearly $F^{SO}_{\overrightarrow{\mu}}$ is a rational function of q and t. The reformulated invariants then can be defined by

$$g_{\overrightarrow{\mu}}(q,t) = \sum_{k|\overrightarrow{\mu}} \frac{\mu(k)}{k} F^{SO}_{\overrightarrow{\mu}/k}(q^k,t^k),$$

where $\mu(k)$ is the Möbius function.

5. Orthogonal Labastida-Mariño-Ooguri-Vafa Conjecture

5.1. **Orthogonal LMOV Conjecture.** Now we can state the main conjecture of this paper, which is the analog of LMOV conjecture for orthogonal Chern-Simons theory.

Conjecture 5.1 (Orthogonal LMOV). *The rational function $g_{\overrightarrow{\mu}}(q,t)\in\mathbb{Q}(q,t)$ has the property that*

$$\frac{\mathfrak{z}_{\overrightarrow{\mu}}(q-q^{-1})^2\cdot[g_{\overrightarrow{\mu}}(q,t)-g_{\overrightarrow{\mu}}(q,-t)]}{2\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha}-q^{-\mu_i^\alpha})} \in \mathbb{Z}[q-q^{-1}][t,t^{-1}].$$

We may write the above (conjectured) polynomial as

$$\frac{\mathfrak{z}_{\overrightarrow{\mu}}(q-q^{-1})^2\cdot[g_{\overrightarrow{\mu}}(q,t)-g_{\overrightarrow{\mu}}(q,-t)]}{2\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha}-q^{-\mu_i^\alpha})} = \sum_{g\in\mathbb{Z}_+/2}\sum_{\beta\in\mathbb{Z}} N_{\overrightarrow{\mu},g,\beta}(q-q^{-1})^{2g}t^{\beta}.$$

The integers $N_{\overrightarrow{\mu},g,\beta}$ (or their linear combinations, depends on a choice of basis) are explained as BPS numbers in string theory [7, 34], and these numbers should coincide with the BPS numbers calculated by the Gromov-Witten theory (see for example [37, 39]). Physicists predict that the Gromov-Witten theory of orientifold are dual to the type-B Chern-Simons theory [7], i.e, the partition functions of these two theories coincide up to some normalization. Thus the integers $N_{\overrightarrow{\mu},g,\beta}$ are conjecturally equal to some linear combinations of intersection numbers on the moduli space of stable maps from curves into un-oriented manifolds. However, a mathematical construction of such moduli space is still lacking.

Remark 5.1. Actually the anti-symmetrization $\frac{g_{\overrightarrow{\mu}}(q,t)-g_{\overrightarrow{\mu}}(q,-t)}{2}$ is not necessary for some knots/links. Thus if we expand $\frac{\mathfrak{z}_{\overrightarrow{\mu}}(q-q^{-1})^2 g_{\overrightarrow{\mu}}(q,t)}{\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha}-q^{-\mu_i^\alpha})}$, then we may get more integer coefficients. Readers may find the proof of most theorems except for some cases of Theorem 1.7 are still valid for $\frac{\mathfrak{z}_{\overrightarrow{\mu}}(q-q^{-1})^2 g_{\overrightarrow{\mu}}(q,t)}{\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha}-q^{-\mu_i^\alpha})}$.

Remark 5.2. Physicists Bouchard-Florea-Mariño [7] and more recently Mariño [34] have similar conjectures for a different partition functions. It seems none of these definitions are equivalent to the definition given here. However, it is pointed out by Mariño that the reformulated invariants may coincide for some examples of torus knots. But we obtain different integer invariants for torus links. Thus relations between these conjectures are still unclear at the present time. It shows that anti-symmetrization process is not necessary for some links and knots. Anyway, in

next subsection, we will leave the integer coefficient invariants of torus knots and links before anti-symmetrization for interested readers to investigate the relationship between the conjecture proposed in [7, 34] and ours.

To describe the behavior of the reformulated invariants near $q = 1$, let $q = e^u$ and embed $\mathbb{Q}(q,t)$ into $\mathbb{Q}(t)((u))$. Denote $val_u(F^{SO}_{\overrightarrow{\mu}})$ the valuation of $F^{SO}_{\overrightarrow{\mu}}$ in the valuation field $\mathbb{Q}(t)((u))$. This valuation is the same as the zero order of the rational function $F^{SO}_{\overrightarrow{\mu}}$ at $q = 1$.

Conjecture 5.2 (Degree). *The valuation $val_u(F^{SO}_{\overrightarrow{\mu}})$ is greater than or equal to $\ell(\overrightarrow{\mu}) - 2$.*

The Conjecture 5.2 is indeed claiming that all the coefficients of lower degree vanish. It is not a consequence of Conjecture 5.1. We will see later that this vanishing is closely related to formulas of Lickorish-Millett type. This kind of degree conjecture is also an important part of the Liu-Peng's paper [32]. We will prove Conjecture 5.2 in Section 8 and 9.

5.2. **Torus Link as Supporting Examples of Main Conjecture.** In this subsection, we verify the orthogonal LMOV conjecture by testing torus links and knots for small partitions.

In addition, several examples of torus links and knots of type $T(2,k)$ suggest that the anti-symmetrization of the reformulated invariants $g_{\overrightarrow{\mu}}(q,t)$ in the Conjecture 5.1 is necessary. In the following, we will denote $q - q^{-1}$ by z for simplicity. We compute the colored Kauffman polynomials for these examples in Section 10 (Appendix). For tables of integer coefficients $N_{\overrightarrow{\mu},g,\beta}$ of these torus links and knots, please refer to Section 10 (Appendix).

Example 1: Take $r = 1$, the torus link $T(2,2k)$ has 2 components.

Case A: Consider the partition $(1),(1)$ for link $T(2,2k)$

Denote $W_{(n)}(unknot)$ by $W_{(n)}$ in the following computations, where $n \in \mathbb{Z}_{\geq 0}$.

It is easy to verify that

$$\begin{aligned}
\mathrm{z}_{(1),(1)} g_{(1),(1)} &= W_{(1),(1)} - W_{(1)}^2 \\
&= q^{2k} sb_{(2)} + q^{-2k} sb_{(1,1)} + t^{-2k} - sb_{(1)}^2 \\
&= \left(\frac{q^{2k+1} + q^{-2k-1}}{q^1 + q^{-1}} - 1 \right) t^2 + \left(\frac{q^{2k-1} + q^{-2k+1}}{q + q^{-1}} - 1 \right) t^{-2} - \left(\frac{q^k - q^{-k}}{q - q^{-1}} \right)^2 + t^{-2k}.
\end{aligned}$$

Thus all the integer invariant numbers $N_{\overrightarrow{\mu},g,\beta} = 0$.

Case B: Consider the partition $(1,1),(1)$ for link $T(2,2k)$:

$$\begin{aligned}
\mathrm{z}_{(1,1),(1)} g_{(1,1),(1)} &= 2(Z_{(1,1),(1)} - Z_{(1),(1)} Z_{(1)} - Z_{(1,1)} Z_{(1)} + Z_{(1)}^3) \\
&= W_{(2),(1)} + W_{(1,1),(1)} + W_{(1)} - 2W_{(1),(1)} W_{(1)} - (W_{(2)} + W_{(1,1)} + 1) W_{(1)} + 2W_{(1)}^3 \\
&= q^{4k} sb_{(3)} + q^{-2k} sb_{(2,1)} + q^{-2k} t^{-2k} sb_{(1)} + q^{2k} sb_{(2,1)} + q^{-4k} sb_{(1,1,1)} + q^{2k} t^{-2k} sb_{(1)} \\
&\quad - 2(q^{2k} sb_{(2)} + q^{-2k} sb_{(1,1)} + t^{-2k}) sb_{(1)} + 2sb_{(1)}^3 - (sb_{(2)} + sb_{(1,1)}) sb_{(1)}.
\end{aligned}$$

It is interesting that the rational function $\frac{(q-q^{-1})^2}{(q-q^{-1})^3}\mathrm{z}_{(1,1),(1)}g_{(1,1),(1)}(q,t)$ is already in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ without anti-symmetrization. We list the first three in this family:

$$
\begin{aligned}
\frac{(q-q^{-1})^2\mathrm{z}_{(1,1),(1)}g_{(1,1),(1)}(T(2,2))}{(q-q^{-1})^3} &= (-t^{-3}+3t^{-1}-3t+t^3)+(t^{-2}-2+t^2)z\\
\frac{(q-q^{-1})^2\mathrm{z}_{(1,1),(1)}g_{(1,1),(1)}(T(2,4))}{(q-q^{-1})^3} &= (-4t^{-5}+4t^{-3}+12t^{-1}-20t+8t^3)\\
&+(4t^{-4}+4t^{-2}-20+12t^2)z\\
&+(-t^{-5}+t^{-3}+3t^{-1}-9t+6t^3)z^2\\
&+(t^{-4}+t^{-2}-9+7t^2)z^3+(-t+t^3)z^4+(-1+t^2)z^5\\
\frac{(q-q^{-1})^2\mathrm{z}_{(1,1),(1)}g_{(1,1),(1)}(T(2,6))}{(q-q^{-1})^3} &= (-9t^{-7}+9t^{-5}-3t^{-3}+45t^{-1}-72t+30t^3)\\
&+(9t^{-6}+27t^{-2}-90+54t^2)z\\
&+(-6t^{-7}+6t^{-5}-t^{-3}+39t^{-1}-93t+55t^3)z^2\\
&+(6t^{-6}+27t^{-2}-114+81t^2)z^3\\
&+(-t^{-7}+t^{-5}+11t^{-1}-47t+36t^3)z^4\\
&+(t^{-6}+9t^{-2}-55+45t^2)z^5\\
&+(t^{-1}-11t+10t^3)z^6+(-12+t^{-2}+11t^2)z^7\\
&+(-t+t^3)z^8+(-1+t^2)z^9.
\end{aligned}
$$

Please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$ after anti-symmetrization.

The conjectural prediction on $g_{(1,1),(1)}$ is also proved in Section 7. Next we compute $g_{(2),(1)}(T(2,2k))$, which will not be covered by any proof in following sections.

Case C: Consider the partition $(2),(1)$ for link $T(2,2k)$

$$
\begin{aligned}
\mathrm{z}_{(2),(1)}g_{(2),(1)} &= 2(Z_{(2),(1)}-Z_{(2)}Z_{(1)})\\
&= (W_{(2),(1)}-W_{(1,1),(1)}+W_{(1)})-W_{(1)}(W_{(2)}-W_{(1,1)}+1)\\
&= (q^{4k}sb_{(3)}+q^{-2k}sb_{(2,1)}+q^{-2k}t^{-2k}sb_{(1)}-q^{2k}sb_{(2,1)}-q^{-4k}sb_{(1,1,1)}-q^{2k}t^{-2k}sb_{(1)}+sb_{(1)})\\
&-sb_{(1)}(sb_{(2)}-sb_{(1,1)}+1)
\end{aligned}
$$

The rational function $\frac{(q-q^{-1})^2}{(q-q^{-1})(q^2-q^{-2})}\mathfrak{z}_{(2),(1)}g_{(2),(1)}(q,t)$ is also in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ without anti-symmetrization. We list the first three in this family:

$$\begin{aligned}
\frac{(q-q^{-1})^2\mathfrak{z}_{(2),(1)}g_{(2),(1)}(T(2,2))}{(q-q^{-1})(q^2-q^{-2})} &= (t^{-3}-t^{-1}-t+t^3)+(-t^{-2}+t^2)z\\
\frac{(q-q^{-1})^2\mathfrak{z}_{(2),(1)}g_{(2),(1)}(T(2,4))}{(q-q^{-1})(q^2-q^{-2})} &= (2t^{-5}-2t^{-3}+2t^{-1}-6t+4t^3)+(-2t^{-4}+2t^{-2}-6+6t^2)z\\
&+(t^{-5}-t^{-3}+t^{-1}-5t+4t^3)z^2+(-t^{-4}+t^{-2}-5+5t^2)z^3\\
&+(-t+t^3)z^4+(-1+t^2)z^5\\
\frac{(q-q^{-1})^2\mathfrak{z}_{(2),(1)}g_{(2),(1)}(T(2,6))}{(q-q^{-1})(q^2-q^{-2})} &= (3t^{-7}-3t^{-5}-t^{-3}+9t^{-1}-18t+10t^3)\\
&+(-3t^{-6}+9t^{-2}-24+18t^2)z\\
&+(4t^{-7}-4t^{-5}-t^{-3}+15t^{-1}-39t+25t^3)z^2\\
&+(-4t^{-6}+15t^{-2}-50+39t^2)z^3\\
&+(t^{-7}-t^{-5}+7t^{-1}-29t+22t^3)z^4\\
&+(-t^{-6}+7t^{-2}-35+29t^2)z^5\\
&+(t^{-1}-9t+8t^3)z^6+(\frac{1}{t^2}-10+9t^2)z^7\\
&+(-t+t^3)z^8+(-1+t^2)z^9.
\end{aligned}$$

Please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$ after anti-symmetrization.

The behavior of $g_{(2),(2)}(T(2,2k);q,t)$ is much different from the above three examples. It is the first example that the multi-cover contribution must be taken into account

Case D: Consider the partition $(2),(2)$ for link $T(2,2k)$

$$\begin{aligned}
\mathfrak{z}_{(2),(2)}g_{(2),(2)} &= 4F_{(2),(2)}(q,t)-2F_{(1),(1)}(q^2,t^2)\\
&= 4Z_{(2),(2)}(q,t)-4Z^2_{(2)}(q,t)-2(Z_{(1),(1)}(q^2,t^2)-4Z^2_{(1)}(q^2,t^2))\\
&= W_{(2),(2)}-2W_{(2),(1,1)}+W_{(1,1),(1)}-W^2_{(2)}-W^2_{(1,1)}+2W_{(2)}W_{(1,1)}\\
&-2W_{(1),(1)}(q^2,t^2)+2W^2_{(1)}(q^2,t^2)\\
&= q^{8k}sb_{(4)}+(1-2q^{4k})sb_{(3,1)}+(q^{4k}+q^{-4k})sb_{(2,2)}+(1-2q^{-4k})sb_{(2,1,1)}+q^{-8k}sb_{(1,1,1,1)}\\
&+(q^{-2k}-2q^{2k}+q^{6k})t^{-2k}sb_{(2)}+(q^{-6k}-2q^{-2k}+q^{2k})t^{-2k}sb_{(1,1)}+(q^{4k}+q^{-4k})t^{-4k}\\
&-(sb_{(2)}-sb_{(1,1)})^2-2(q^{4k}sb_{(2)}(q^2,t^2)+q^{-4k}sb_{(1,1)}(q^2,t^2)+t^{-4k})+2sb^2_{(1)}(q^2,t^2).
\end{aligned}$$

The rational function $\frac{(q-q^{-1})^2}{(q^2-q^{-2})^2}\mathfrak{z}_{(2),(2)}g_{(2),(2)}(q,t)$ is not in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ and anti-symmetrization is necessary here. Please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$.

Case E: Consider the partition $(3),(1)$ for link $T(2,2k)$

$$\begin{aligned}
\mathfrak{z}_{(3),(1)}g_{(3),(1)} &= 3(Z_{(3),(1)}-Z_{(3)}Z_{(1)})\\
&= (W_{(3),(1)}-W_{(2,1),(1)}+W_{(1,1,1),(1)})-W_{(1)}(W_{(3)}-W_{(2,1)}+W_{(1,1,1)})\\
&= (q^{6k}-1)sb_{(4)}+(q^{-2k}-q^{4k})sb_{(3,1)}+(q^{2k}-q^{-4k})sb_{(2,1,1)}+(q^{-6k}-1)sb_{(1,1,1,1)}\\
&+(q^{-4k}-q^{2k})t^{-2k}sb_{(2)}+(q^{4k}-q^{-2k})t^{-2k}sb_{(1,1)}.
\end{aligned}$$

The rational function $\frac{(q-q^{-1})^2}{(q^3-q^{-3})(q-q^{-1})} \mathrm{z}_{(3),(1)} g_{(3),(1)}(q,t)$ is not in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ and anti-symmetrization is necessary here. Please see Section 10 for the table of integers $N_{\overrightarrow{\mu},g,\beta}$ after anti-symmetrization.

Example 2: The torus knots $T(2,k)$, where k is an odd integer.

Case A: Consider the partition $(1,1)$ for link $T(2,k)$

The following calculation provides an example of the case proved in Section 7. We have

$$\begin{aligned}
\mathrm{z}_{(1,1)} g_{(1,1)} &= 2g_{(1,1)} \\
&= W_{(2)} + W_{(1,1)} - W_{(1)}^2(q,t) + 1 \\
&= t^{-2k}(q^{2k}sb_{(4)} - q^{-2k}sb_{(3,1)} + q^{-4k}sb_{(2,2)} + q^{-3k}t^{-k}sb_{(2)} - q^{-5k}t^{-k}sb_{(1,1)} + q^{-4k}t^{-2k}) \\
&+ t^{-2k}(q^{4k}sb_{(2,2)} - q^{2k}sb_{(2,1,1)} + q^{-2k}sb_{(1,1,1,1)} + q^{5k}t^{-k}sb_{(2)} - q^{3k}t^{-k}sb_{(1,1)} + q^{4k}t^{-2k}) \\
&- t^{-2k}(q^{k}sb_{(2)} - q^{-k}sb_{(1,1)} + t^{-k})^2 + 1.
\end{aligned}$$

It is interesting that The rational function $\frac{(q-q^{-1})^2}{(q-q^{-1})^2} \mathrm{z}_{(1,1)} g_{(1,1)}(q,t)$ is already in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ without anti-symmetrization.

We list the Trefoil knot in this family:

$$\begin{aligned}
\frac{(q-q^{-1})^2 \mathrm{z}_{(1,1)} g_{(1,1)}(T(2,3))}{(q-q^{-1})^2} &= (t^{-12} + 6t^{-10} - 33t^{-8} + 52t^{-6} - 33t^{-4} + 6t^{-2} + 1) \\
&+ (36t^{-11} - 132t^{-9} + 180t^{-7} - 108t^{-5} + 24t^{-3})z \\
&+ (36t^{-12} - 103t^{-10} + 76t^{-8} + 18t^{-6} - 32t^{-4} + 5t^{-2})z^2 \\
&+ (105t^{-11} - 377t^{-9} + 453t^{-7} - 207t^{-5} + 26t^{-3})z^3 \\
&+ (105t^{-12} - 350t^{-10} + 341t^{-8} - 87t^{-6} - 10t^{-4} + t^{-2})z^4 \\
&+ (112t^{-11} - 450t^{-9} + 494t^{-7} - 165t^{-5} + 9t^{-3})z^5 \\
&+ (112t^{-12} - 441t^{-10} + 440t^{-8} - 110t^{-6} - t^{-4})z^6 \\
&+ (54t^{-11} - 275t^{-9} + 286t^{-7} - 66t^{-5} + t^{-3})z^7 \\
&+ (54t^{-12} - 274t^{-10} + 274t^{-8} - 54t^{-6})z^8 \\
&+ (12t^{-11} - 90t^{-9} + 91t^{-7} - 13t^{-5})z^9 \\
&+ (12t^{-12} - 90t^{-10} + 90t^{-8} - 12t^{-6})z^{10} \\
&+ (t^{-11} - 15t^{-9} + 15t^{-7} - t^{-5})z^{11} \\
&+ (t^{-12} - 15t^{-10} + 15t^{-8} - t^{-6})z^{12} \\
&+ (-t^{-9} + t^{-7})z^{13} + (-t^{-10} + t^{-8})z^{14}.
\end{aligned}$$

Please see Section 10 for the table of integers $N_{\overrightarrow{\mu},g,\beta}$ after anti-symmetrization.

Case B: Consider the partition (2) for link $T(2,k)$

The function

$$
\begin{aligned}
\mathrm{z}_{(2)} g_{(2)} &= 2g_{(2)} \\
&= W_{(2)} - W_{(1,1)} - W_{(1)}(q^2, t^2) + 1 \\
&= t^{-2k}[(q^{2k} - q^{-2k})sb_{(3,1)} + (q^{-4k} - q^{4k} - q^{2k} + q^{-2k})sb_{(2,2)} + (q^{2k} - q^{-2k})sb_{(2,1,1)} \\
&+ ((q^{-3k} - q^{5k})t^{-k} - q^{2k} + q^{-2k})sb_{(2)} + ((-q^{-5k} + q^{3k})t^{-k} + q^{2k} - q^{-2k})sb_{(1,1)} \\
&+ (q^{-4k} - q^{4k} - 1)t^{-2k} - q^{2k} + q^{-2k}] + 1.
\end{aligned}
$$

The rational function $\frac{(q-q^{-1})^2}{(q^2-q^{-2})}\mathrm{z}_{(2)} g_{(2)}(q,t)$ is not in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$. Please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$ after anti-symmetrization.

Example 3: Take $r = 1$, the torus link $T(3, 3k)$ has 3 components.

Consider the partition $(2),(1),(1)$ for link $T(3, 3k)$.

Denote $W_{(1),(1)}(T(2, 2k))$ simply by $W_{(1),(1)}$ in the following computations.

The function

$$
\begin{aligned}
\mathrm{z}_{(2),(1),(1)} g_{(2),(1),(1)} &= 2g_{(2),(1),(1)} \\
&= W_{(2),(1),(1)} - W_{(1,1),(1),(1)} - W_{(1),(1)}W_{(2)} + W_{(1),(1)}W_{(1,1)} \\
&- 2W_{(2),(1)}W_{(1)} + 2W_{(1,1),(1)}W_{(1)} + 2W_{(2)}W_{(1)}^2 - 2W_{(1,1)}W_{(1)}^2 \\
&= q^{10k}sb_{(4)} + (2q^{2k} - q^{6k})sb_{(3,1)} + (q^{-2k} - q^{2k})sb_{(2,2)} \\
&+ (q^{-6k} - 2q^{-2k})sb_{(2,1,1)} - q^{-10k}sb_{(1,1,1,1)} \\
&+ (3 - 2q^{4k})t^{-2k}sb_{(2)} + (2q^{-4k} - 3)t^{-2k}sb_{(1,1)} + (q^{-2k} - q^{2k})t^{-4k} \\
&- (q^{2k}sb_{(2)} + q^{-2k}sb_{(1,1)} - 2sb_{(1)}^2 + t^{-2k})(sb_{(2)} - sb_{(1,1)}) \\
&+ 2(-q^{4k}sb_{(3)} + (q^{2k} - q^{-2k})sb_{(2,1)} + q^{-4k}sb_{(1,1,1)} + (q^{2k} - q^{-2k})t^{-2k}sb_{(1)})sb_{(1)}
\end{aligned}
$$

The rational function $\frac{(q-q^{-1})^2}{(q^2-q^{-2})(q-q^{-1})^2}\mathrm{z}_{(2),(1),(1)} g_{(2),(1),(1)}(q,t)$ is in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$.

We list the first two in this family

$$
\begin{aligned}
\frac{(q-q^{-1})^2 \mathrm{z}_{(2),(1),(1)} g_{(2),(1),(1)}(T(3,3))}{(q-q^{-1})^2(q^2-q^{-2})} &= (-t^{-4}+4t^{-2}+2-12t^2+7t^4)+(2t^{-3}-2t^{-1}-10t+10t^3)z \\
&+ (1-6t^2+5t^4)z^2+(-6t+6t^3)z^3 \\
&+ (-t^2+t^4)z^4+(-t+t^3)z^5 \\
\frac{(q-q^{-1})^2 \mathrm{z}_{(2),(1),(1)} g_{(2),(1),(1)}(T(3,6))}{(q-q^{-1})(q^2-q^{-2})} &= (-2t^{-8}-4t^{-6}+22t^{-4}-48t^{-2}+146-204t^2+90t^4) \\
&+ (16t^{-5}-48t^{-3}+176t^{-1}-336t+192t^3)z \\
&+ (-t^{-8}-2t^{-6}+15t^{-4}-68t^{-2}+361-650t^2+345t^4)z^2 \\
&+ (12t^{-5}-68t^{-3}+452t^{-1}-1036t+640t^3)z^3 \\
&+ (2t^{-4}-38t^{-2}+398-950t^2+588t^4)z^4 \\
&+ (2t^{-5}-38t^{-3}+494t^{-1}-1406t+948t^3)z^5 \\
&+ (-10t^{-2}+239-780t^2+551t^4)z^6 \\
&+ (-10t^{-3}+286t^{-1}-1056t+780t^3)z^7 \\
&+ (-t^{-2}+80-377t^2+298t^4)z^8 \\
&+ (-t^{-3}+91t^{-1}-467t+377t^3)z^9 \\
&+ (14-106t^2+92t^4)z^{10}+(15t^{-1}-121t+106t^3)z^{11} \\
&+ (1-16t^2+15t^4)z^{12}+(t^{-1}-17t+16t^3)z^{13} \\
&+ (-t^2+t^4)z^{14}+(-t+t^3)z^{15}
\end{aligned}
$$

After anti-symmetrization we obtain the table of the integers $N_{\overrightarrow{\mu},g,\beta}$ for $\overrightarrow{\mu}=(2),(1),(1)$.

Untill now, we have seen the orthogonal LMOV conjecture is valid for the knots $T(2,k)$ and $T(3,3k)$ with small number k.

In fact, we can prove them for arbitrary $k \in \mathbb{Z}_{>0}$.

For instance, we investigate torus knot $T(2,k)$ for odd integer number k with partition (2). We can express

$$\mathrm{z}_{(2)}(g_{(2)}(q,t)-g_{(2)}(q,-t))/2$$

in terms of pb's instead of sb's. After simplification, we have

$$
\begin{aligned}
\frac{\mathrm{z}_{(2)}(g_{(2)}(q,t)-g_{(2)}(q,-t))}{2} &= t^{-2k}\frac{t-t^{-1}}{q-q^{-1}}\Big(\frac{(q^{2k}-q^{-2k})}{(q-q^{-1})(q^3-q^{-3})}(-(t^2+t^{-2})(q^{2k}+q^{-2k}-q^2-q^{-2}) \\
&+ (q^4+q^{-4})(q^{2k}+q^{-2k})-q^4-2-q^{-4}) \\
&+ \frac{t^{-k}(q^{4k}-q^{-4k})}{q^2-q^{-2}}(-t(q^{k+1}-q^{-k-1})+t^{-1}(q^{k-1}-q^{-k+1}))\Big),
\end{aligned}
$$

It is easy to see that the rational function

$$\frac{(q-q^{-1})^2}{2(q^2-q^{-2})}(g_{(2)}(q,t)-g_{(2)}(q,-t))$$

is in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ by a tedious discussion on the residue of k modulo 6. Actually, all these examples can be proved in such a manner.

6. Formulas of Lickorish-Millett Type

The Skein relations of Kauffman polynomials are

(1) $< \mathcal{L}_+ > - < \mathcal{L}_- > = z(< \mathcal{L}_{||} > - < \mathcal{L}_= >)$,
where $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{L}_{||}$ and $\mathcal{L}_=$ stand for positive crossing, negative crossing, vertical resolution and horizontal resolution respectively.

(2) $< \mathcal{L}^{+kink} > = t < \mathcal{L} >, \quad < \mathcal{L}^{-kink} > = t^{-1} < \mathcal{L} >$.

The variable z is our $q - q^{-1}$ in previous sections, and the Kauffman polynomials are given by

$$K_{\mathcal{L}}(z,t) = t^{-2lk(\mathcal{L})} < \mathcal{L} >$$

with the normalization $K_{\bigcirc}(z,t) = 1$ for the unknot $\bigcirc$. In terms of quantum group invariants, we have

$$W^{SO}_{(1)^L}(\mathcal{L}) = (1 + \frac{t - t^{-1}}{z}) < \mathcal{L} > .$$

The Kauffman polynomials admit the expansions

$$K_{\mathcal{L}}(z,t) = \sum_{g\geq 0} \widetilde{p}^{\mathcal{L}}_{g+1-L}(t) z^{g+1-L}$$

and

$$< \mathcal{L} > = \sum_{g\geq 0} p^{\mathcal{L}}_{g+1-L}(t) z^{g+1-L}$$

with respect to variable z. The classical Lickorish-Millett Formula [28] reads

$$\widetilde{p}^{\mathcal{L}}_{1-L}(t) = t^{-2lk(\mathcal{L})}(t - t^{-1})^{L-1} \prod_{\alpha=1}^{L} p_0^{\mathcal{K}_\alpha}(t)$$

and so

$$p^{\mathcal{L}}_{1-L}(t) = (t - t^{-1})^{L-1} \prod_{\alpha=1}^{L} p_0^{\mathcal{K}_\alpha}(t),$$

which give a concrete description of $\widetilde{p}^{\mathcal{L}}_{1-L}(t)$, the coefficient of the lowest degree terms of $K_{\mathcal{L}}(z,t)$, in terms of invariants of the sub-knots $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_L$ of $\mathcal{L}$. In the following theorem, we provide explicit formulas for $p^{\mathcal{L}}_{2-L}(t)$ and $p^{\mathcal{L}}_{3-L}(t)$, which are regarded as higher Lickorish-Millett relations. These formulas can be proved purely by skein relations. Through resolving intersections at different link components, it is not hard to prove the following. Also, these formulas can be directly deduced from the Conjecture 5.2 (See Section 7). In [20], Kanenobu got some relationships(non-explicit) between these terms.

Theorem 6.1. *Let $\mathcal{L}_{1,2}$ be the sub-link of $\mathcal{L}$ which composed of components K_1 and K_2. The coefficients $p^{\mathcal{L}}_{2-L}(t)$ and $p^{\mathcal{L}}_{3-L}(t)$ are given by the formulas*

$$p^{\mathcal{L}}_{2-L}(t) = (L-1)(t - t^{-1})^{L-2} p_0^{\mathcal{K}_1}(t) \cdots p_0^{\mathcal{K}_L}(t) + (t - t^{-1})^{L-1}(p_1^{\mathcal{K}_1}(t) p_0^{\mathcal{K}_2}(t) \cdots p_0^{\mathcal{K}_L}(t) + perm);$$

$$\begin{aligned} p^{\mathcal{L}}_{3-L}(t) = \binom{L-1}{2} (t - t^{-1})^{L-3} p_0^{\mathcal{K}_1}(t) \cdots p_0^{\mathcal{K}_L}(t) + (t - t^{-1})^{L-2}(p_1^{\mathcal{L}_{1,2}}(t) p_0^{\mathcal{K}_3}(t) \cdots p_0^{\mathcal{K}_L}(t) + perm) \\ - (L-2)(t - t^{-1})^{L-1}(p_2^{\mathcal{K}_1}(t) p_0^{\mathcal{K}_2}(t) \cdots p_0^{\mathcal{K}_L}(t) + perm). \end{aligned}$$

Proof. The formulas in the theorem are obvious when $L = 1$, and the formula for $p^{\mathcal{L}}_{3-L}(t)$ is also valid for $L = 2$. We prove the theorem by induction. Let $\mathcal{L}$ be a link with $L + 1$ components. The main idea is using skein relations at the intersection points of different components of the $\mathcal{L}$ until the component $\mathcal{K}_{L+1}$ split from the link.

First we perform the skein relation at the crossings between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ until there is no intersection between them. We need do skein relation $(n^+_{1,L+1}+n^-_{1,L+1})/2$ times, where $n^+_{1,L+1}$ and $n^-_{1,L+1}$ denote the number of positive and negative crossings between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ respectively. Thus the linking number between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ is $lk(\mathcal{L}_{1,L+1}) = (n^+_{1,L+1} - n^-_{1,L+1})/2$.

From the calculation, one can see that do the skein relation at a positive crossing will lead similar result. Thus without loss of generality, we can assume $n^-_{1,L+1} > 0$ and do the skein relation at a negative crossing first:

$$< \mathcal{L}_+ > - < \mathcal{L}_- >= z(< \mathcal{L}_{(1\|L+1),2,\cdots,L} > - < \mathcal{L}_{(1=L+1),2,\cdots,L} >),$$

where $\mathcal{L}_-$ is the original link $\mathcal{L}$, $(1\|L+1)$(resp.$(1=L+1)$) is the new knot component derived from $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ by taking vertical (resp.horizontal) lines as its resolution at the intersection of $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ in the new link $\mathcal{L}_{(1\|L+1),2,\cdots,L}$ (resp.$\mathcal{L}_{(1=L+1),2,\cdots,L}$). Both $\mathcal{L}_{(1\|L+1),2,\cdots,L}$ and $\mathcal{L}_{(1=L+1),2,\cdots,L}$ are of L-components, while $\mathcal{L}_+$ is the link obtained simply by changing the sign of the chosen crossing, thus has the same $L+1$ components as the original link $\mathcal{L} = \mathcal{L}_-$.

Taking the few leading terms in the skein relation formula

$$\begin{aligned}&(p^{\mathcal{L}_+}_{-L}(t)z^{-L} + p^{\mathcal{L}_+}_{1-L}(t)z^{1-L} + p^{\mathcal{L}_+}_{2-L}(t)z^{2-L}) - (p^{\mathcal{L}}_{-L}(t)z^{-L} + p^{\mathcal{L}}_{1-L}(t)z^{1-L} + p^{\mathcal{L}}_{2-L}(t)z^{2-L})\\ &= z(p^{\mathcal{L}_{(1\|L+1),2,\cdots,L}}_{1-L}(t)z^{1-L} - p^{\mathcal{L}_{(1=L+1),2,\cdots,L}}_{1-L}(t)z^{1-L})\end{aligned}$$

and comparing the coefficients, we have

1) $p^{\mathcal{L}_+}_{-L}(t) = p^{\mathcal{L}}_{-L}(t)$ (this one gives the formula for $p^{\mathcal{L}}_{1-L}$, which we don't use.)
2) $p^{\mathcal{L}_+}_{1-L}(t) = p^{\mathcal{L}}_{1-L}(t)$
3) $p^{\mathcal{L}_+}_{2-L}(t) - p^{\mathcal{L}}_{2-L}(t) = p^{\mathcal{L}_{(1\|L+1),2,\cdots,L}}_{1-L} - p^{\mathcal{L}_{(1=L+1),2,\cdots,L}}_{1-L}$

By the Lickorish-Millett Formula,

$$p^{\mathcal{L}_{(1\|L+1),2,\cdots,L}}_{1-L} = (t-t^{-1})^{L-1}p^{\mathcal{K}_2}_0(t)\cdots p^{\mathcal{K}_L}_0(t)p^{\mathcal{K}_{(1\|L+1)}}_0(t)$$

and

$$p^{\mathcal{L}_{(1=L+1),2,\cdots,L}}_{1-L} = (t-t^{-1})^{L-1}p^{\mathcal{K}_2}_0(t)\cdots p^{\mathcal{K}_L}_0(t)p^{\mathcal{K}_{(1=L+1)}}_0(t),$$

where $\mathcal{K}_{(1\|L+1)}$ (resp.$\mathcal{K}_{(1=L+1)}$) is the knot derived from the sub-link $\mathcal{L}_{1,L+1}$ by taking vertical (resp.horizontal) lines as its resolution at the chosen crossing. Thus

$$p^{\mathcal{L}_+}_{2-L} - p^{\mathcal{L}}_{2-L} = (t-t^{-1})^{L-1}p^{\mathcal{K}_2}_0\cdots p^{\mathcal{K}_L}_0(p^{\mathcal{K}_{(1\|L+1)}}_0 - p^{\mathcal{K}_{(1=L+1)}}_0). \tag{6.1}$$

We play a trick here to find the expression of $p^{\mathcal{K}_{(1\|L+1)}}_0$ and $p^{\mathcal{K}_{(1=L+1)}}_0$. Consider the sub-link $\mathcal{L}_{1,L+1}$ of $\mathcal{L}$, which only have two components $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$. Then do the skein relation at exactly the same crossing as we did in the original link $\mathcal{L}$. The same argument lead to

$$p^{(\mathcal{L}_{1,L+1})_+}_1 - p^{\mathcal{L}_{1,L+1}}_1 = p^{\mathcal{K}_{(1\|L+1)}}_0 - p^{\mathcal{K}_{(1=L+1)}}_0.$$

which substitute back gives

$$p^{\mathcal{L}_+}_{2-L} - p^{\mathcal{L}}_{2-L} = (t-t^{-1})^{L-1}p^{\mathcal{K}_2}_0\cdots p^{\mathcal{K}_L}_0(p^{(\mathcal{L}_{1,L+1})_+}_1 - p^{\mathcal{L}_{1,L+1}}_1) \tag{6.2}$$

In the above equation, $p^{\mathcal{L}}_{2-L}$ is expressed in terms of invariants of $\mathcal{L}_+$ and some simple terms. Then we do skein relations $\mathcal{L}_+$ at other intersection points between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ until these two components become unlinked. We cancel all the middle state in this procedure, and finally we reached

$$p^{\mathcal{L}^{(1)}}_{2-L} - p^{\mathcal{L}}_{2-L} = (t-t^{-1})^{L-1}p^{\mathcal{K}_2}_0\cdots p^{\mathcal{K}_L}_0(p^{\mathcal{L}^{(1)}_{1,L+1}}_1 - p^{\mathcal{L}_{1,L+1}}_1). \tag{6.3}$$

Here $\mathcal{L}^{(1)}$ is the final state of $\mathcal{L}$, in which $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ are unlinked. $\mathcal{L}^{(1)}_{1,L+1}$ is the corresponding final state of $\mathcal{L}_{1,L+1}$ under the same procedure of skein relations. In $\mathcal{L}^{(1)}_{1,L+1}$, the two components $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ are unlinked too, i.e, $\mathcal{L}^{(1)}_{1,L+1}$ is the disjoint union of two knots $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$, $W^{SO}_{(1),(1)}(\mathcal{L}^{(1)}_{1,L+1}) = W^{SO}_{(1)}(\mathcal{K}_1)W^{SO}_{(1)}(\mathcal{K}_{L+1})$.

By definition of Kauffman polynomials, $W^{SO}(\mathcal{L}) = (1+\frac{t-t^{-1}}{z}) < \mathcal{L} >$ for all links $\mathcal{L}$, we have

$$< \mathcal{L}^{(1)}_{1,L+1} >= (1+\frac{t-t^{-1}}{z}) < \mathcal{K}_1 >< \mathcal{K}_{L+1} >$$

Up to the third leading terms,

$$p_{-1}^{\mathcal{L}^{(1)}_{1,L+1}}z^{-1} + p_0^{\mathcal{L}^{(1)}_{1,L+1}} + p_1^{\mathcal{L}^{(1)}_{1,L+1}}z = (1+\frac{t-t^{-1}}{z})(p_0^{\mathcal{K}_1} + p_1^{\mathcal{K}_1}z + p_2^{\mathcal{K}_1}z^2)(p_0^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_{L+1}}z + p_2^{\mathcal{K}_{L+1}}z^2)$$

and comparing the coefficients:

1) $p_{-1}^{\mathcal{L}^{(1)}_{1,L+1}} = (t-t^{-1})p_0^{\mathcal{K}_1}p_0^{\mathcal{K}_{L+1}}$,
2) $p_0^{\mathcal{L}^{(1)}_{1,L+1}} = p_0^{\mathcal{K}_1}p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_0^{\mathcal{K}_1}p_1^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_1}p_0^{\mathcal{K}_{L+1}})$,
3) $p_1^{\mathcal{L}^{(1)}_{1,L+1}} = p_0^{\mathcal{K}_1}p_1^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_1}p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_0^{\mathcal{K}_1}p_2^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_1}p_1^{\mathcal{K}_{L+1}} + p_2^{\mathcal{K}_1}p_0^{\mathcal{K}_{L+1}})$.

In summary, we have the following formula now

$$\begin{aligned}
&p_{2-L}^{\mathcal{L}^{(1)}} - p_{2-L}^{\mathcal{L}}\\
&= (t-t^{-1})^{L-1}p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L}p_1^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L-1}p_1^{\mathcal{K}_1}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L}p_0^{\mathcal{K}_{L+1}}\\
&+ (t-t^{-1})^{L}p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L}p_2^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L}p_1^{\mathcal{K}_1}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L}p_1^{\mathcal{K}_{L+1}}\\
&+ (t-t^{-1})^{L}p_2^{\mathcal{K}_1}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L}p_0^{\mathcal{K}_{L+1}} - (t-t^{-1})^{L-1}p_1^{\mathcal{L}_{1,L+1}}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L}.
\end{aligned}$$

Next perform all the above procedures for the link $\mathcal{L}^{(1)}$ to the final state $\mathcal{L}^{(1)(2)}$ in which the components $\mathcal{K}_2$ and $\mathcal{K}_{L+1}$ are unlinked.

Repeat this process totally L times until $\mathcal{K}_{L+1}$ is unlinked to the sub-link $\mathcal{L}_{1,\cdots,L}$ of $\mathcal{L}$, the result is given by

$$\begin{aligned}
&p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} - p_{2-L}^{\mathcal{L}}\\
&= L(t-t^{-1})^{L-1}p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L}p_1^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L-1}(p_1^{\mathcal{K}_1}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm)p_0^{\mathcal{K}_{L+1}}\\
&+ L(t-t^{-1})^{L}p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L}p_2^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L}(p_1^{\mathcal{K}_1}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm)p_1^{\mathcal{K}_{L+1}}\\
&+ (t-t^{-1})^{L}(p_2^{\mathcal{K}_1}p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm)p_0^{\mathcal{K}_{L+1}} - (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,L+1}}p_0^{K_2}\cdots p_0^{\mathcal{K}_L} + perm).
\end{aligned}$$

As the link $\mathcal{L}^{(1)\cdots(L)}$ is the disjoint union of the sub-link $\mathcal{L}_{1,\cdots,L}$ of $\mathcal{L}$ and the knot $\mathcal{K}_{L+1}$, $W^{SO}_{(1)^{L+1}}(\mathcal{L}^{(1)\cdots(L)}) = W^{SO}_{(1)^L}(\mathcal{L}_{1,\cdots,L})W^{SO}_{(1)}(\mathcal{K}_{L+1})$. Again, this can be rewrite into the form

$$< \mathcal{L}^{(1)\cdots(L)} >= (1+\frac{t-t^{-1}}{z}) < \mathcal{L}_{1,\cdots,L} >< \mathcal{K}_{L+1} >$$

as $lk(\mathcal{L}^{(1)\cdots(L)}) = lk(\mathcal{L}_{1,\cdots,L})$. Up to third leading terms,

$$\begin{aligned}
&p_{-L}^{\mathcal{L}^{(1)\cdots(L)}}z^{-L} + p_{-L}^{\mathcal{L}^{(1)\cdots(L)}}z^{1-L} + p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}}z^{2-L}\\
&= (1+\frac{t-t^{-1}}{z})(p_{1-L}^{\mathcal{L}_{1,\cdots,L}}z^{1-L} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}}z^{2-L} + p_{3-L}^{\mathcal{L}_{1,\cdots,L}}z^{3-L})(p_0^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_{L+1}}z + p_2^{\mathcal{K}_{L+1}}z^2)
\end{aligned}$$

and comparing the coefficients

1) $p_{-L}^{\mathcal{L}^{(1)\cdots(L)}} = (t-t^{-1})p_{1-L}^{\mathcal{L}_{1,\cdots,L}}p_0^{\mathcal{K}_{L+1}}$,

2) $p_{1-L}^{\mathcal{L}^{(1)\cdots(L)}} = p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}})$,

3) $p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} = p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_2^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}} p_1^{\mathcal{K}_{L+1}} + p_{3-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}})$.

We now can finish the proof by induction. Be careful that our link $\mathcal{L}$ has $L+1$ components. The sub-link $\mathcal{L}_{1,\cdots,L}$ has L components and by induction

$$p_{2-L}^{\mathcal{L}_{1,\cdots,L}} = (L-1)(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} + (t-t^{-1})^L (p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm),$$

so

$$\begin{aligned}
&p_{2-(L+1)}^{\mathcal{L}}(t) = p_{1-L}^{\mathcal{L}^{(1)\cdots(L)}}\\
&= p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}})\\
&= L(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})^L (p_1^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} + perm).
\end{aligned}$$

This finishes the proof of the first part of the theorem.

Now we have sufficient results to prove the second part. We have seen that

$$\begin{aligned}
p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} &= p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\cdots,L}} p_2^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\cdots,L}} p_1^{\mathcal{K}_{L+1}} + p_{3-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}})\\
&= L(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_1^{\mathcal{K}_{L+1}} + (L-1)(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}}\\
&+ (t-t^{-1})^{L-1}(p_1^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} + perm) p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})^L p_0^{\mathcal{K}_1}\cdots p_0^{K_L} p_2^{\mathcal{K}_{L+1}}\\
&+ (t-t^{-1})^L (p_1^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} + perm) p_1^{\mathcal{K}_{L+1}} + (t-t^{-1}) p_{3-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}}.
\end{aligned}$$

Combined with the expression of $p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} - p_{2-L}^{\mathcal{L}}$,we got an expression of $p_{2-L}^{\mathcal{L}}$ in terms of sub-links

$$\begin{aligned}
p_{2-L}^{\mathcal{L}} &= (L-1)(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} - (L-1)(t-t^{-1})^L p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_2^{\mathcal{K}_{L+1}}\\
&- (t-t^{-1})^L (p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm) p_0^{\mathcal{K}_{L+1}}\\
&+ (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,L+1}} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm) + (t-t^{-1}) p_{3-L}^{\mathcal{L}_{1,\cdots,L}} p_0^{\mathcal{K}_{L+1}}.
\end{aligned}$$

Since the sub-link $\mathcal{L}_{1,\cdots,L}$ of $\mathcal{L}$ contains L components, by induction we have

$$\begin{aligned}
p_{3-L}^{\mathcal{L}_{1,\cdots,L}}(t) &= \binom{L-1}{2}(t-t^{-1})^{L-3} p_0^{\mathcal{K}_1}(t)\cdots p_0^{\mathcal{K}_L}(t) + (t-t^{-1})^{L-2}(p_1^{\mathcal{L}_{1,2}}(t) p_0^{\mathcal{K}_3}(t)\cdots p_0^{\mathcal{K}_L}(t) + perm)\\
&- (L-2)(t-t^{-1})^{L-1}(p_2^{\mathcal{K}_1}(t) p_0^{\mathcal{K}_2}(t)\cdots p_0^{\mathcal{K}_L}(a) + perm).
\end{aligned}$$

Here the permutation only involves the first L components. Later, when computing the invariants of $\mathcal{L}$, the permutations will also include the $L+1$'th component. As the content is self-evident, we will not mention this issue in the future. Substitute the above induction formula to the expression of $\mathcal{L}$, the proof of the second part of the theorem is finished:

$$\begin{aligned}
p_{2-L}^{\mathcal{L}} &= (L-1)(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} - (L-1)(t-t^{-1})^L p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_2^{\mathcal{K}_{L+1}}\\
&- (t-t^{-1})^L (p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm) p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,L+1}} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm)\\
&+ [\binom{L-1}{2}(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} + (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,2}} p_0^{\mathcal{K}_3}\cdots p_0^{\mathcal{K}_L} + perm)\\
&- (L-2)(t-t^{-1})^L (p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + perm)] p_0^{\mathcal{K}_{L+1}}\\
&= \binom{L}{2}(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,2}} p_0^{\mathcal{K}_3}\cdots p_0^{\mathcal{K}_{L+1}} + perm)\\
&- (L-1)(t-t^{-1})^L (p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_{L+1}} + perm).
\end{aligned}$$

□

7. The Proof of the Conjecture for Column Diagram

In the last section, we provide two formulas of Lickorish-Millett type. In general, similar computation leads to expressions of $p_n^{\mathcal{L}}(t)$ in terms of invariants of sub-links of $\mathcal{L}$. Each additional component of $\mathcal{L}$ gives rise to two such relations; thus we expect there should be $2L-2$ such relations, i.e., all the $p_n^{\mathcal{L}}(t)$'s for $1-L \leq n \leq L$ should be able to be described by sub-links of $\mathcal{L}$.

When the index n increases, the expression become messy. To give a unified treatment, we formulate the problem in terms of the partition function $Z_{CS}^{SO}(\mathcal{L};q,t)$ and free energy $F^{SO}(\mathcal{L};q,t)$. Recall that we write

$$Z_{CS}^{SO}(\mathcal{L};q,t) = 1 + \sum_{\overrightarrow{\mu}\neq\overrightarrow{0}} Z_{\overrightarrow{\mu}}^{SO} pb_{\overrightarrow{\mu}}$$

and

$$F^{SO}(\mathcal{L};q,t) = \sum_{\overrightarrow{\mu}\neq\overrightarrow{0}} F_{\overrightarrow{\mu}}^{SO} pb_{\overrightarrow{\mu}}.$$

Where $\overrightarrow{\mu} = (\mu^1,\cdots,\mu^L)$ for partitions $\mu^1,\cdots,\mu^L$. In this section, we mainly focus on the situation when all μ^i are column like partitions. We look at the first situation that all μ^i are partitions 1 now. We simply denote such $\overrightarrow{\mu}$ by $(1)^L = (1),\cdots,(1)$, since the partition of 1 is unique and there is no ambiguity. The coefficients $Z_{(1)^L}^{SO} = W_{(1)^L}^{SO}$.

Let Δ be a subset of the set $[L] := \{1,\cdots,L\}$. Write $\mathcal{L}_\Delta$ for the sub-link of $\mathcal{L}$ composed only by the components with labels in Δ. For example, when $\Delta = \{1,2\}$, $\mathcal{L}_\Delta$ is the link $\mathcal{L}_{1,2}$ discussed in the previous section. We also denote by Δ the partition $\overrightarrow{\mu} = (\mu^1,\cdots,\mu^L)$ such that $\mu^i = (1)$ if $i \in \Delta$, and 0 otherwise. The convention in the definition of quantum group invariants is $W_\Delta^{SO}(\mathcal{L}) := W_{(1)^{|\Delta|}}^{SO}(\mathcal{L}_\Delta)$. The formula (4.12) then can be written as

$$F_{(1)^L}^{SO}(\mathcal{L}) = \sum_{r=1}^{L} \frac{(-1)^{r-1}}{r} \sum_{\Delta_1,\cdots,\Delta_r} \prod_{i=1}^{r} W_{\Delta_i}^{SO}(\mathcal{L}) \tag{7.1}$$

where the second summation is over all nonempty subsets $\Delta_1,\cdots,\Delta_r$ which form a partition of the set $[L]$. We have seen that $F_{(1)^L}^{SO}(\mathcal{L}) \in \mathbb{Q}(t)((z))$ for $z = q - q^{-1}$ has an expansion

$$F_{(1)^L}^{SO}(\mathcal{L}) = \sum_{i\geq -L} a_i(t) z^i.$$

Conjecture 5.2 predict that $val_z F_{(1)^L}^{SO}(\mathcal{L}) \geq L-2$, i.e, $a_{-L} = a_{1-L} = \cdots = a_{L-3} = 0$. We now prove $a_{-L} = a_{1-L} = a_{2-L} = 0$ by the classical Lichorish-Millett theorem and the two formulas derived in last section.

Theorem 7.1. *Expand $F_{(1)^L}^{SO}(\mathcal{L})$ as above, then we have the vanishing result $a_{-L} = a_{1-L} = a_{2-L} = 0$ if $L \geq 3$. In other words, $val_u(F_{(1)^L}^{SO}(\mathcal{L})) = val_z(F_{(1)^L}^{SO}(\mathcal{L})) \geq 3-L$. In the case $L=2$, the second formula in Theorem 6.1 is empty, thus we only have $a_{-2} = a_{-1} = 0$ and $val_z(F_{(1)^2}^{SO}(\mathcal{L})) \geq 0$.*

Proof. We prove the theorem for a_{1-L} when $L \geqslant 2$ by calculating (7.1). The proof for a_{-L} and a_{2-L} are similar and we leave them to the reader.

$$W_\Delta^{SO}(\mathcal{L}) \cong (1 + \frac{t-t^{-1}}{z})(p_{1-|\Delta|}^{\mathcal{L}_\Delta} z^{1-|\Delta|} + p_{2-|\Delta|}^{\mathcal{L}_\Delta} z^{2-|\Delta|} + p_{3-|\Delta|}^{\mathcal{L}_\Delta} z^{3-|\Delta|}) (\operatorname{mod} z^{3-|\Delta|})$$

Denote by $[z^n]f$ the coefficient of z^n in $f \in \mathbb{Q}(t)((z))$.

$$a_{1-L} = \sum_{r=1}^{L} \frac{(-1)^{r-1}}{r} \sum_{\Delta_1,\cdots,\Delta_r} [z^{1-L}](1 + \frac{t-t^{-1}}{z})^r \prod_{i=1}^{r} < \mathcal{L}_{\Delta_i} >$$

For each possible collection $\Delta_1, \cdots, \Delta_r$,

$$
\begin{aligned}
&[z^{1-L}](1+\frac{t-t^{-1}}{z})^r \prod_{i=1}^{r} < \mathcal{L}_{\Delta_i} > \\
&= r(t-t^{-1})^{r-1} \prod_{i=1}^{r} p_{1-|\Delta_i|}^{\mathcal{L}_{\Delta_i}} + (t-t^{-1})^r \sum_{i=1}^{r} p_{1-|\Delta_1|}^{\mathcal{L}_{\Delta_1}} \cdots \widehat{p_{1-|\Delta_i|}^{\mathcal{L}_{\Delta_i}}} \cdots p_{1-|\Delta_r|}^{\mathcal{L}_{\Delta_r}} \cdot p_{2-|\Delta_i|}^{\mathcal{L}_{\Delta_i}} \\
&= r(t-t^{-1})^{L-1} \prod_{i=1}^{r} \prod_{j\in\Delta_i} p_0^{K_j}(t) + (t-t^{-1})^{L-1} \sum_{k=1}^{r} \prod_{\substack{i=1\\ i\neq k}}^{r} \prod_{j\in\Delta_i} p_0^{K_j}(t) \cdot [(|\Delta_k|-1) \prod_{j\in\Delta_k} p_0^{K_j} \\
&+ (t-t^{-1}) \sum_{l\in\Delta_k} p_1^{K_l} \prod_{\substack{j\in\Delta_k\\ j\neq l}} p_0^{K_j}] \\
&= L(t-t^{-1})^{L-1} \prod_{\alpha=1}^{L} p_0^{K_\alpha}(t) + (t-t^{-1})^L \sum_{j=1}^{L} p_1^{K_j} \prod_{\substack{i=1\\ i\neq j}}^{L} p_0^{K_i}
\end{aligned}
$$

has the same contribution. We need to count the number of these collections. Let Λ be a partition of L of length r, the number of collections $\{\Delta_1, \cdots, \Delta_r\}$ with $\{|\Delta_1|, \cdots, |\Delta_r|\}$ equal to the partition Λ is given by $\frac{r!}{|Aut\Lambda|} \cdot \frac{L!}{\Lambda_1! \cdots \Lambda_r!}$, hence

$$
a_{1-L} = (L(t-t^{-1})^{L-1} \prod_{\alpha=1}^{L} p_0^{K_\alpha}(t) + (t-t^{-1})^L \sum_{j=1}^{L} p_1^{K_j} \prod_{\substack{i=1\\ i\neq j}}^{L} p_0^{K_i}) \cdot \sum_{\Lambda\vdash L} \frac{(-1)^{\ell(\Lambda)-1}\ell(\Lambda)!}{\ell(\Lambda)|Aut\Lambda|} \cdot \frac{L!}{\Lambda!}
$$

which is zero by the following Lemma 7.2 as $L \geq 2$. □

Lemma 7.2. *Assume $d_\alpha \geq 1$ for $\alpha = 1, 2, \cdots, L$ and the sum $d = d_1 + \cdots + d_L$ is strictly greater than 1 (i.e., if all $d_i = 1$, then we assume $L > 1$), then*

$$
\sum_{\overrightarrow{\lambda}\vdash\overrightarrow{d}} \frac{(-1)^{\ell(\overrightarrow{\lambda})-1}\ell(\overrightarrow{\lambda})!}{\ell(\overrightarrow{\lambda})|Aut\overrightarrow{\lambda}| \prod_{\alpha=1}^{L} \prod_{j=1}^{\ell(\lambda^\alpha)} \lambda_j^\alpha!} = 0. \tag{7.2}
$$

Proof. Let $\overrightarrow{t} = (t_1, \cdots, t_L)$ and $|\overrightarrow{t}| = t_1 + \cdots + t_L$ in the trivial equality

$$
|\overrightarrow{t}| = \log(\exp(|\overrightarrow{t}|)) = \log(1 + \sum_{n=1}^{+\infty} \frac{|\overrightarrow{t}|^n}{n!})
$$

so we have

$$
t_1 + \cdots + t_L = \log(1 + \sum_{\substack{\overrightarrow{\beta}\in\mathbb{Z}_{\geq 0}^L\\ \overrightarrow{\beta}\neq 0}} \frac{\overrightarrow{t}^{\overrightarrow{\beta}}}{\overrightarrow{\beta}!})
$$

where we have adopt the notation $\overrightarrow{t}^{\overrightarrow{\beta}} = \prod_{\alpha=1}^{L} t_\alpha^{\beta_\alpha}$ and $\overrightarrow{\beta}! = \prod_{\alpha=1}^{L} (\beta_\alpha!)$. Expand the logarithm

$$
t_1 + \cdots + t_L = \sum_{\overrightarrow{\beta}\in\mathbb{Z}_{\geq 0}^L} \overrightarrow{t}^{\overrightarrow{\beta}} \sum_{\overrightarrow{\lambda}\vdash\overrightarrow{\beta}} \frac{(-1)^{\ell(\overrightarrow{\lambda})-1}}{\ell(\overrightarrow{\lambda})|Aut\overrightarrow{\lambda}| \prod_{\alpha=1}^{L} \lambda^\alpha!}.
$$

and comparing the coefficients of the term $t_1^{d_1}\cdots t_L^{d_L}$ gives the vanishing formula. □

We remark that the vanishing of a_{1-L} and a_{2-L} also imply the formulas for $p_{2-L}^{\mathcal{L}}$ and $p_{3-L}^{\mathcal{L}}$ proved in last section. The approach in the previous section has the merit that it produces explicit expressions, while the statement in terms of free energy can give a uniform treatment to contain all the relations of Lickorish-Millett type, as in the following theorem.

Theorem 7.3. *Under the same notations as above, then we have the vanishing result $a_{-L} = a_{1-L} = \cdots = a_{L-3} = 0$. In other words, $val_z(F^{SO}_{(1)^L}(\mathcal{L})) \geq L-2$. Indeed, we have*

$$(q-q^{-1})^{2-L}F^{SO}_{(1)^L}(\mathcal{L}) \in \mathbb{Z}[t,t^{-1}][q-q^{-1}].$$

As a corollary, Conjecture 5.1 is true for partitions $\overrightarrow{\mu} = (1,1,\cdots,1)$.

Proof. We prove the theorem by induction. When $L=1$, $\mathcal{L}$ is a knot, $F^{SO}_{(1)^1}(\mathcal{L}) = W^{SO}_{(1)}(\mathcal{L}) = (1+\frac{t-t^{-1}}{z}) < \mathcal{L} >= (1+\frac{t-t^{-1}}{z})K_{\mathcal{L}}$ for the Kauffman polynomial of $\mathcal{L}$ obviously has z-valuation equal to $-1 = L-2$. The theorem thus holds for knots.

Now Assume $\mathcal{L}$ is a link with $L>1$ components $\mathcal{K}_1,\cdots,\mathcal{K}_L$. We first deal with the simple case when $\mathcal{L}$ is the disjoint union of $\mathcal{K}_\alpha$'s. Then for any partition $\Delta_1,\cdots,\Delta_r$ of the set $[L]$, the product $\prod_{i=1}^r W^{SO}_{\Delta_i}(\mathcal{L}) = \prod_{\alpha=1}^L W^{SO}_{(1)}(\mathcal{K}_\alpha)$ is independent of the partition. Again let Λ be a partition of L of length r, the number of collections $\{\Delta_1,\cdots,\Delta_r\}$ with $\{|\Delta_1|,\cdots,|\Delta_r|\}$ equal to the partition Λ is given by $\frac{r!}{|Aut\Lambda|}\cdot\frac{L!}{\Lambda_1!\cdots\Lambda_r!}$, hence

$$F^{SO}_{(1)^L}(\mathcal{L}) = \prod_{\alpha=1}^L W^{SO}_{(1)}(\mathcal{K}_\alpha)\cdot\sum_{\Lambda\vdash L}\frac{(-1)^{\ell(\Lambda)-1}\ell(\Lambda)!L!}{\ell(\Lambda)|Aut\Lambda|\Lambda!} = 0.$$

There is another way to see this directly. For if the link $\mathcal{L}$ is the disjoint union of $\mathcal{K}_\alpha$'s, then the free energy $F^{SO}(\mathcal{L}, pb(z_1),\cdots,pb(z_L))$ is the sum of the free energy $F^{SO}(\mathcal{K}_\alpha; pb(z_\alpha))$. The expansion of such a sum $F^{SO}(\mathcal{L})$ with respect to $pb_{\overrightarrow{\mu}}$ does not contain terms of the form $\prod_{\alpha=1}^L p_1(z_\alpha)$. Thus the theorem is true for links of the type of disjoint union.

Finally, consider the Skein relation

$$< \mathcal{L}_+ > - < \mathcal{L}_- >= z(< \mathcal{L}_{||} > - < \mathcal{L}_= >),$$

where $< \mathcal{L}_+ >$ and $< \mathcal{L}_- >$ are two links coincide everywhere except at one crossing P between two different components $\mathcal{K}_a$ and $\mathcal{K}_b$ of the link $\mathcal{L}$ for $1 \leq a < b \leq L$. The link $< \mathcal{L}_{||} >$ (resp. $< \mathcal{L}_= >$) is the link by replacing the crossing P by two parallel vertical (resp. horizontal) lines. Both $< \mathcal{L}_{||} >$ and $< \mathcal{L}_= >$ have $L-1$ components. Let's compute the difference

$$F^{SO}_{(1)^L}(\mathcal{L}_+) - F^{SO}_{(1)^L}(\mathcal{L}_-) = \sum_{r=1}^L\frac{(-1)^{r-1}}{r}\sum_{\Delta_1,\cdots,\Delta_r}(\prod_{i=1}^r W^{SO}_{\Delta_i}(\mathcal{L}_+) - \prod_{i=1}^r W^{SO}_{\Delta_i}(\mathcal{L}_-)).$$

The summation is again over all partitions $\Delta_1,\cdots,\Delta_r$ of the set $[L]$. An important observation is that $\prod_{i=1}^r W^{SO}_{\Delta_i}(\mathcal{L}_+) - \prod_{i=1}^r W^{SO}_{\Delta_i}(\mathcal{L}_-) = 0$ if a and b are not in the same set Δ_i for some i, because in this situation the sub-links $\mathcal{L}_{+,\Delta_i}$ coincide to the sub-links $\mathcal{L}_{-,\Delta_i}$. In particular, this is the

case if $r = L$. The above difference can be simplified

$$\begin{aligned}
&F^{SO}_{(1)^L}(\mathcal{L}_+) - F^{SO}_{(1)^L}(\mathcal{L}_-)\\
&= \sum_{r=1}^{L-1} \frac{(-1)^{r-1}}{r} \sum_{i=1}^{r} \sum_{\Delta_1,\cdots,\Delta_r; a,b\in\Delta_i} (W^{SO}_{\Delta_i}(\mathcal{L}_+) - W^{SO}_{\Delta_i}(\mathcal{L}_-)) \prod_{j=1,j\neq i}^{r} W^{SO}_{\Delta_j}(\mathcal{L}_+)\\
&= \sum_{r=1}^{L-1} \frac{(-1)^{r-1}}{r} \sum_{i=1}^{r} \sum_{\Delta_1,\cdots,\Delta_r; a,b\in\Delta_i} z\cdot (W^{SO}_{\Delta_i}(\mathcal{L}_{||}) - W^{SO}_{\Delta_i}(\mathcal{L}_=)) \prod_{j=1,j\neq i}^{r} W^{SO}_{\Delta_j}(\mathcal{L}_+)\\
&= z\cdot (F^{SO}_{(1)^{L-1}}(\mathcal{L}_{||}) - F^{SO}_{(1)^{L-1}}(\mathcal{L}_=)).
\end{aligned}$$

By induction, both $z^{2-(L-1)}\cdot F^{SO}_{(1)^{L-1}}(\mathcal{L}_{||})$ and $z^{3-L}\cdot F^{SO}_{(1)^{L-1}}(\mathcal{L}_=)$ are in the ring $\mathbb{Z}[t,t^{-1}][z]$, thus if the theorem is true for link $\mathcal{L}_+$ if and only if it is true for link $\mathcal{L}_-$.

For a general link $\mathcal{L}$ which is not necessarily a disjoint union, changing crossings between different components of $\mathcal{L}$ respectively until it becomes a disjoint union of L knots. As the theorem is true for disjoint union, it is true for $\mathcal{L}$. □

The whole results of Section 6 can be view as application of Theorem 7.3 combined with some combinatorial identities like Lemma 7.2.

To study the cases of partitions with more boxes, we first develop the cabling technique. Let β be a braid of which the closure is the link $\mathcal{L}$. For each $\overrightarrow{d}\in\mathbb{Z}^L_+$, denote $\beta_{\overrightarrow{d}}$ the braid obtained by cabling the k-th strand of β to d_α parallel ones if it in the α-th component of $\mathcal{L}$. The partition function of $\mathcal{L}$ and the Kauffman polynomials are related by the following Lemma.

Lemma 7.4. *Assume β is of writhe zero on every components, then the partition function of $\mathcal{L}$ is related to the Kauffman polynomial of the cabling link by*

$$W^{SO}_{(1)^d}(\beta_{\overrightarrow{d}}) = \sum_{\overrightarrow{A}\in\widehat{Br}_{\overrightarrow{d}}} \chi_{\overrightarrow{A}}(\mathrm{id}) W^{SO}_{\overrightarrow{A}}(\mathcal{L};q,t) = \overrightarrow{d}!\cdot Z^{SO}_{((1^{d_1}),\cdots,(1^{d_L}))}(\mathcal{L}),$$

where $d=\sum_{\alpha=1}^{L} d_\alpha$ and $\overrightarrow{d}! = \mathfrak{z}_{((1^{d_1}),\cdots,(1^{d_L}))} = d_1!\cdots d_L!$.

Proof. Take a β of zero writhe on every component, the cabling link $\beta_{\overrightarrow{d}}$ is also of zero writhe on every component, and the quantum group invariants $W_{\overrightarrow{A}}$ are equal to the trace of

$$\beta_{\overrightarrow{d}}\cdot (p_{A^1}\otimes\cdots\otimes p_{A^L}) \tag{7.3}$$

in the Birman-Murakami-Wenzl algebra C_M for $M = d_1r_1+\cdots+d_Lr_L$ and p_{A^α} is the minimal idempotents in C_{d_α} corresponding to the irreducible representation numbered by the partition A^α. Apparently, each p_{A^α} should appear r_i times in the above tensor. However, the naturality of the universal $\mathcal{R}$-matrices plus the trace property will move all p_{A^α} to the same strand and thus one p_{A^α} for each $\alpha=1,2,\cdots,L$ is enough.

The expansion coefficients $Z^{SO}_{(1^{d_1},\cdots,1^{d_L})}(\mathcal{L})$ of the partition function can be calculated directly

$$\begin{aligned}
\overrightarrow{d}!\cdot Z^{SO}_{(1^{d_1},\cdots,1^{d_L})}(\mathcal{L}) &= \sum_{\overrightarrow{A}\in\widehat{Br}_{\overrightarrow{d}}} \chi_{\overrightarrow{A}}(\mathrm{id}) W^{SO}_{\overrightarrow{A}}(\mathcal{L};q,t)\\
&= \sum_{\overrightarrow{A}\in\widehat{Br}_{\overrightarrow{d}}} \chi_{\overrightarrow{A}}(\mathrm{id})\mathrm{tr}_{V^M}(\beta_{\overrightarrow{d}}\cdot(p_{A^1}\otimes\cdots\otimes p_{A^L}))\\
&= \mathrm{tr}_{V^M}(\beta_{\overrightarrow{d}})\\
&= W^{SO}_{(1)^d}(\beta_{\overrightarrow{d}};q,t).
\end{aligned}$$

We have used the fact that for a semi-simple algebra, the dimension of an irreducible representation $\chi_{A^i}(\mathrm{id})$ is the same as the multiplicity of A^i in the semi-simple decomposition of the algebra, so

$$\sum_{\overrightarrow{A}\in\widehat{Br}_{\overrightarrow{d}}} \chi_{\overrightarrow{A}}(\mathrm{id})(p_{A^1}\otimes\cdots\otimes p_{A^L}) = \mathrm{id}$$

in the third equality. □

Remark 7.1. A similar formula holds for HOMFLY polynomials and can be proved in the same way.

Theorem 7.5. *For partitions $\overrightarrow{\mu} = (\mu^1,\cdots,\mu^L)\in\mathcal{P}^L$ such that $\mu^\alpha = (1,1,\cdots,1)\vdash d_\alpha$ for each $\alpha = 1,\cdots,L$, we have*

$$\overrightarrow{d}!(q-q^{-1})^{2-d}\cdot F_{\overrightarrow{\mu}}(\mathcal{L},q,t)\in\mathbb{Z}[t,t^{-1}][q-q^{-1}].$$

In particular, the Conjecture 5.1 (Orthgonal LMOV Conj.) is valid for such column like partitions.

Proof. We will use the symbol $(1)^{\overrightarrow{d}}$ to denote the partition $\overrightarrow{\mu}$ in the theorem. Let β be a braid whose closure is the link $\mathcal{L}$ with zero writhe. Let $\beta_{\overrightarrow{d}}$ be the cabling braid as in section 3. The calculation in Lemma 7.4 in fact provide that

$$Z^{SO}_{(1)^{\overrightarrow{d}}}(\mathcal{L}) = \frac{1}{\overrightarrow{d}!}W^{SO}_{(1)^d}(\beta_{\overrightarrow{d}}),$$

which reduce the situation back to the Kauffman case. A more careful observation is the following cabling equality

$$F^{SO}_{(1)^{\overrightarrow{d}}}(\mathcal{L}) = \frac{1}{\overrightarrow{d}!}F^{SO}_{(1)^d}(\beta_{\overrightarrow{d}}),$$

which, together with Theorem 7.3, finishes the proof.

We now prove the cabling equality by comparing both sides. The LHS equal to

$$\begin{aligned}
&\sum_{r=1}^{d}\frac{(-1)^{r-1}}{r}\sum_{\overrightarrow{A_1},\cdots,\overrightarrow{A_r}} Z^{SO}_{\overrightarrow{A_1}}(\mathcal{L})\cdots Z^{SO}_{\overrightarrow{A_r}}(\mathcal{L})\\
&=\sum_{r=1}^{d}\frac{(-1)^{r-1}}{r}\sum_{\overrightarrow{A_1},\cdots,\overrightarrow{A_r}}\frac{W^{SO}_{(1)^{||\overrightarrow{A_1}||}}(\beta_{|\overrightarrow{A_1}|})\cdots W^{SO}_{(1)^{||\overrightarrow{A_r}||}}(\beta_{|\overrightarrow{A_r}|})}{\overrightarrow{A_1}!\cdots\overrightarrow{A_r}!},
\end{aligned}$$

where the summation is over all partitions $(\overrightarrow{A_1},\cdots,\overrightarrow{A_r})$ of length of the partition $(1)^{\overrightarrow{d}}$. As each $\overrightarrow{A_i}$ must be of the form $((1^{a_i^1}),(1^{a_i^2}),\cdots,(1^{a_i^L}))$ such that $\sum_{i=1}^{r}a_i^\alpha = d_\alpha$ for every $\alpha = 1,2,\cdots,L$. Then the symbols $|\overrightarrow{A_i}| = (a_i^1,a_i^2,\cdots,a_i^L)$, $||\overrightarrow{A_i}|| = a_i^1+a_i^2+\cdots+a_i^L$ and $\overrightarrow{A_i}! = a_i^1!a_i^2!\cdots a_i^L!$ as in the introduction. Again β is the braid with zero writhe on every components representing the link $\mathcal{L}$, and $\beta_{|\overrightarrow{A_i}|}$ is the cabling link.

The RHS equal to

$$\frac{1}{\overrightarrow{d}!}\sum_{r=1}^{d}\frac{(-1)^{r-1}}{r}\sum_{\Delta_1,\cdots,\Delta_r}\prod_{i=1}^{r}W^{SO}_{\Delta_i}(\beta_{\overrightarrow{d}})$$

for $\Delta_1,\cdots,\Delta_r$ are non-empty sets which form a partition of the set $[d]$. Each Δ_i can be further decompose into a partition $\Xi_i^1,\Xi_i^2,\cdots,\Xi_i^L$, such that elements in Ξ_i^α labelling the components in $\beta_{\overrightarrow{d}}$ arising from the cabling of the αth component of $\mathcal{L}$. Write $a_i^\alpha = |\Xi_i^\alpha|$ for the number of

elements in Ξ_i^α, which can be zero. Then the vectors $\overrightarrow{A_i}$'s defined by $\overrightarrow{A_i} = ((1^{a_i^1}), (1^{a_i^2}), \cdots, (1^{a_i^L}))$ become one term in the summation appear in the LHS. Furthermore, for each fixed such $\overrightarrow{A_i}$'s, there are $\prod_{\alpha=1}^{L} \frac{d_\alpha!}{a_1^\alpha! \cdots a_r^\alpha!}$ possible partition sets Ξ_i^α's. The equality holds. □

8. The Case of Rows Implies the Conjecture

In this section, we discuss the case for a general partition $\overrightarrow{\mu}$, and reduce it to the case of rectangular ones.

We first define an equivalence relation on BMW algebra C_n: two elements $x, y \in C_n$ are equivalent, denoted by $x \sim y$, if $\mathrm{tr}(xz) = \mathrm{tr}(yz)$ for all central elements $z \in C_n$. Obviously, if two elements x and y are conjugate, say if there exist an invertible element $g \in C_n$, such that $gxg^{-1} = y$, then $x \sim y$. As the algebra C_n is semi-simple, two idempotents p_1 and p_2 are equivalent, if and only if they give isomorphic representations of C_n.

Let p_λ be a minimal path idempotent in C_n. Denote by $m_\mu = \sum_\lambda \chi_\lambda(\gamma_\mu) p_\lambda$, and also regard this as an element in the Grothendieck group of representations of the Birman-Murakami-Wenzl algebra. The branching rule [4] for Birman-Murakami-Wenzl algebra is

$$p_\lambda \otimes 1 = \sum_{\lambda'} p_{\lambda'},$$

where the summation is over all partitions λ' which is either add one box to λ or remove one box from λ. As the characters $\chi_A(\gamma_\mu)$ of Brauer algebra are all integers, repeated using the branching rule lead the decomposition of the tensor product of minimal idempotents:

$$m_{(\mu_1)} \otimes m_{(\mu_2)} \otimes \cdots \otimes m_{(\mu_\ell)} \sim \sum_A b_A p_A, \tag{8.1}$$

where the summation is over all possible partitions A and the multiplicity b_A are all integers. Furthermore, the integers b_A are uniquely determined by this equivalence relation, by multiplying both sides the minimal central idempotents π_A of C_n.

Lemma 8.1. *The integers $b_A = \chi_A(\gamma_\mu)$ for the characters of Brauer algebras.*

Proof. As the Birman-Murakami-Wenzl algebras are deformations of the Brauer algebras, they share the same branching rules. Specialize (8.1) to the Brauer algebras by fixing $x = 1 + \frac{t-t^{-1}}{q-q^{-1}}$ and let t and q goes to 1, and using the isomorphism $Br_n \cong \mathrm{End}_{\mathfrak{so}(2N+1)}(V^{\otimes n})$ for $x = 2N+1$, we get

$$\widetilde{m}_{(\mu_1)} \otimes \widetilde{m}_{(\mu_2)} \otimes \cdots \otimes \widetilde{m}_{(\mu_\ell)} \sim \sum_A b_A \widetilde{p}_A, \tag{8.2}$$

where $\widetilde{m}_{(\mu_i)} = \sum_{A \in \widehat{Br}_{\mu_i}} \chi_A(\gamma_{(\mu_i)}) \widetilde{p}_A$ and $\widetilde{p}_A$ is a minimal idempotent in $\mathrm{End}_{\mathfrak{so}(2N+1)}(V^{\otimes n})$. Regard (8.2) as an equality in the Grothendieck group of (finite dimensional representations) of the Lie

group $SO(2N+1)$, and the character is given by

$$\begin{aligned}
&\prod_{i=1}^{\ell}[\sum_{h=0}^{[\frac{\mu_i}{2}]}\sum_{\lambda\vdash\mu_i-2h}\chi_\lambda(\gamma_{(\mu_i)})sb_\lambda(z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N)]\\
&=\prod_{i=1}^{\ell}p_{\mu_i}(z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N)\\
&=p_\mu(z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N)\\
&=\sum_{h=0}^{[\frac{|\mu|}{2}]}\sum_{\lambda\vdash|\mu|-2h}\chi_\lambda(\gamma_\mu)sb_\lambda(z_{-N},z_{1-N},\cdots,z_{-1},z_0,z_1,\cdots,z_{N-1},z_N).
\end{aligned}$$

Thus the two elements $\widetilde{m}_{(\mu_1)}\otimes\widetilde{m}_{(\mu_2)}\otimes\cdots\otimes\widetilde{m}_{\mu_\ell}$ and $\widetilde{m}_\mu$ equal in the Grothendieck group of $SO(2N+1)$, which determines the integers $b_A=\chi_A(\gamma_\mu)$, i.e, we have

$$m_{(\mu_1)}\otimes m_{(\mu_2)}\otimes\cdots\otimes m_{(\mu_\ell)}\sim m_\mu.$$

□

Let $\overrightarrow{\ell}=(\ell_1,\cdots,\ell_L)$, and let $\mathcal{L}_{\overrightarrow{\ell}}$ be closure of the cabling braid $\beta_{\overrightarrow{\ell}}$, which is obtained by cabling the α-th component of β into ℓ_α parallel ones. Then we have

$$\begin{aligned}
\mathrm{z}_{\overrightarrow{\mu}}\cdot Z_{\overrightarrow{\mu}}(\mathcal{L})&=\sum_{\overrightarrow{A}}\chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\mu}})\mathrm{tr}(\beta_{\overrightarrow{\ell}}\cdot p_{\overrightarrow{A}})\\
&=\mathrm{tr}(\beta_{\overrightarrow{\ell}}\cdot m_{\overrightarrow{\mu}})\\
&=\mathrm{tr}[\beta_{\overrightarrow{\ell}}\cdot\bigotimes_{\alpha=1}^{L}(m_{(\mu_1^\alpha)}\otimes\cdots\otimes m_{(\mu_{\ell_\alpha}^\alpha)})]\\
&=(\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell_\alpha}\mu_i^\alpha)\cdot Z_{(\mu_1^1),(\mu_2^1),\cdots,(\mu_{\ell_1}^1)(\mu_1^2),(\mu_2^2),\cdots,(\mu_{\ell_2}^2),\dots,(\mu_1^L),(\mu_2^L),\cdots,(\mu_{\ell_L}^L)}(\mathcal{L}_{\overrightarrow{\ell}})
\end{aligned}$$

and

$$\mathrm{z}_{\overrightarrow{\mu}}\cdot F_{\overrightarrow{\mu}}(\mathcal{L})=(\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell_\alpha}\mu_i^\alpha)\cdot F_{(\mu_1^1),(\mu_2^1),\cdots,(\mu_{\ell_1}^1),(\mu_1^2),(\mu_2^2),\cdots,(\mu_{\ell_2}^2),\dots,(\mu_1^L),(\mu_2^L),\cdots,(\mu_{\ell_L}^L)}(\mathcal{L}_{\overrightarrow{\ell}}).\tag{8.3}$$

Now the partition $(\mu_1^1),(\mu_2^1),\cdots,(\mu_{\ell_1}^1),(\mu_1^2),(\mu_2^2),\cdots,(\mu_{\ell_2}^2),\dots,(\mu_1^L),(\mu_2^L),\cdots,(\mu_{\ell_L}^L)\in\mathcal{P}^{|\ell(\overrightarrow{\mu})|}$ has the property that each component is of length one. In particular, it is rectangular, and we have the following theorem.

Theorem 8.2. *The Conjecture 5.1 is true for all partition $\overrightarrow{\mu}$, if and only if it is true for rectangular $\overrightarrow{\mu}$, if and only if it is true for $\overrightarrow{\mu}=(\mu^1,\cdots,\mu^L)$ such that each $\mu^\alpha=(d_\alpha)$ is of length one.*

(8.3) together with Proposition 9.2 implies Conjecture 5.2 (Degree Conj.), i.e, the degree estimate at $q=1$ is valid for all partitions $\overrightarrow{\mu}$.

Theorem 8.3. *Conjecture 5.2 is true for all links and all partitions.*

Theorem 8.3 implies that Conjecture 5.1 is "true at $q=1$" (Theorem 1.6), i.e., the left hand side of Conjecture 5.1 is regular at $q-1$. The situation at other roots of unity seems to be more difficult. There are some torus knots and links examples verified in Section 5, which can be treated as the conjecture at roots of unity besides 1.

9. Estimation of Degree

Call a partition $\lambda \vdash n$ rectangular, if the Young diagram of $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is rectangular, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_\ell$. A rectangular partition is determined by its length ℓ and its size n.

Let $\delta_n = \sigma_1\sigma_2\cdots\sigma_{n-1}$ be a braid in B_n. Let ℓ be an integer dividing n and write $a = n/\ell$, then the braid $(\delta_n)^\ell$ is associated to the rectangular partition $\lambda = (a, a, \cdots, a) \vdash n$. It is easy to see that $h((\delta_n)^n)$ is in the center of C_n. Let A be a partition of $n - 2f$ for some integer f, and let π_A be a minimal central idempotent in C_n, and let p_A be a minimal idempotent such that $p_A\pi_A = p_A$. Under the isomorphism

$$C_n \cong \bigoplus_{A\in\widehat{Br}_n} M_{d_A\times d_A}(\mathbb{C}),$$

the product $h((\delta_n)^n)\cdot\pi_A$ is a scaler matrix at the block corresponding to A, and zero at other places. In Section 3, we know that $h((\delta_n)^n)\cdot\pi_A = q^{\kappa_A}t^{-2f}\pi_A$, which implies that the eigenvalues of $h((\delta_n)^\ell)\cdot\pi_A$ are either 0 or $q^{\kappa_A/a}t^{-2f/a}$ times n-th roots of unity. We conclude that $\mathrm{tr}(h((\delta_n)^\ell)\cdot p_A) = b_A\cdot q^{\kappa_A/a}t^{-2f/a}$ for some rational number b_A. Taking the specialization $q, t \to 1$, we obtain the value $b_A = \chi_A(\gamma_\lambda)$ for the character χ_A of Brauer algebra.

Now we compute $Z_{\overrightarrow{\lambda}}(\mathcal{L}, q, t)$ for rectangular partition $\overrightarrow{\lambda}$. Write $\overrightarrow{\lambda} = (\lambda^1, \cdots, \lambda^L)$ such that $\lambda^\alpha = (a_\alpha, \cdots, a_\alpha) = (a_\alpha{}^{\ell_\alpha})$ for each $\alpha = 1, 2, \cdots, L$. Our goal in this section is to estimate the u degree of $F_{\overrightarrow{\lambda}}(\mathcal{L};, q, t)$ for $u = \log q$ and rectangular partition $\overrightarrow{\lambda}$.

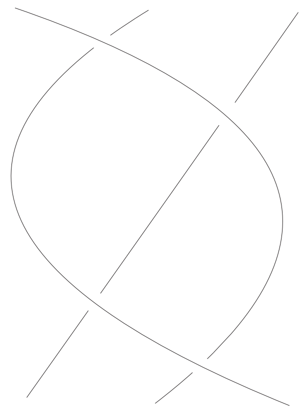
Definition 9.1. Let $\overrightarrow{\tau} = (\tau_1, \cdots, \tau_L) \in \mathbb{C}^L$ be a vector, define the framing dependent link invariants $W_{\overrightarrow{A}}(\mathcal{L}, q, t, \overrightarrow{\tau}) := W_{\overrightarrow{A}}(\mathcal{L}, q, t)q^{\sum_{\alpha=1}^{L}\kappa_{A^\alpha}\tau_\alpha}t^{\sum_{\alpha=1}^{L}|A^\alpha|\tau_\alpha}$, and the framing dependent partition function by

$$Z^{SO}_{CS}(\mathcal{L}; q, t, \overrightarrow{\tau}) = \sum_{\overrightarrow{\mu}\in\mathcal{P}^L}\frac{pb_{\overrightarrow{\mu}}}{z_{\overrightarrow{\mu}}}\cdot\sum_{\overrightarrow{A}\in\widehat{Br}_{|\overrightarrow{\mu}|}}\chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\mu}})W^{SO}_{\overrightarrow{A}}(\mathcal{L}; q, t, \overrightarrow{\tau}). \tag{9.1}$$

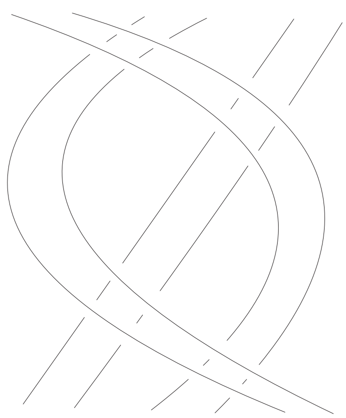
Similarly we define the free energy $F^{SO}(\mathcal{L}; q, t, \overrightarrow{\tau}) = \log Z^{SO}_{CS}(\mathcal{L}; q, t, \overrightarrow{\tau})$ and the coefficients $F^{SO}_{\overrightarrow{\mu}}(\mathcal{L}; q, t, \overrightarrow{\tau})$ and $W^{SO}_{\overrightarrow{\mu}}(\mathcal{L}; q, t, \overrightarrow{\tau})$ as before, replacing the link invariants by the framing dependent invariants. The specialization $\overrightarrow{\tau} = 0$ gives the framing independent invariants.

We compute the partition functions at the special values $\tau_\alpha = w_\alpha + \frac{1}{a_\alpha}$ for $w_\alpha \in \mathbb{Z}$, by taking a braid $\beta(\overrightarrow{w})$ with writhe number w_α on each component $\mathcal{K}_\alpha$ of $\mathcal{L}$. Let $\mathcal{L}^{twist}_{\overrightarrow{n},\overrightarrow{w}}$ be the closure of the product of the cabling braid $\beta_{\overrightarrow{n}}(\overrightarrow{w})$ of $\beta(\overrightarrow{w})$ and the braid $(\omega_{\lambda^1}\otimes\cdots\otimes\omega_{\lambda^L})$, where $n_\alpha = a_\alpha\ell_\alpha$ and $\omega_{\lambda^\alpha} = (\delta_{n_\alpha})^{\ell_\alpha}$. The diagrams below provide an example to illustrate the twisted cabling process in the case $a = 2$, $\ell = 1$ and $n = a\ell = 2$. Suppose $\mathcal{L}$ is a braid in Picture a, which represents a knot with writhe number $w = 4$. Picture b is obtained by cabling each component into two strands. The twist ω_λ is then as in the bottom of Picture c, which add a crossing to

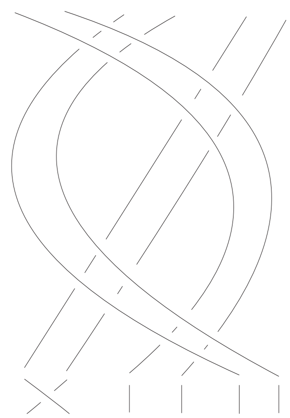
Picture b. The final twisted cabling link $\mathcal{L}^{twist}_{\overrightarrow{n},\overrightarrow{w}}$ is the closure of the braid in Picture d.



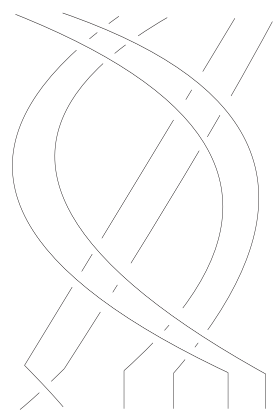
Picture a



Picture b



Picture c



Picture d

The link $\mathcal{L}^{twist}_{\overrightarrow{n},\overrightarrow{w}}$ is of $\ell(\overrightarrow{\lambda}) = \ell_1 + \ell_2 + \cdots + \ell_L$ components, and there are ℓ_i components of writhe $w_\alpha a_\alpha^2 + a_\alpha - 1$.

$$
\begin{aligned}
Z^{SO}_{\overrightarrow{\lambda}}(\mathcal{L}; q, t, \overrightarrow{\tau}) &= \frac{1}{\mathfrak{z}_{\overrightarrow{\lambda}}} \cdot \sum_{\overrightarrow{A} \in \widehat{Br}_{|\overrightarrow{\lambda}|}} \chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\lambda}}) W^{SO}_{\overrightarrow{A}}(\mathcal{L}; q, t, \overrightarrow{\tau}) \\
&= \frac{1}{\mathfrak{z}_{\overrightarrow{\lambda}}} \cdot \mathrm{tr}[\beta_{\overrightarrow{n}}(\overrightarrow{w}) \cdot \sum_{\overrightarrow{A} \in \widehat{Br}_{|\overrightarrow{\lambda}|}} \chi_{\overrightarrow{A}}(\gamma_{\overrightarrow{\lambda}}) q^{\sum_{\alpha=1}^{L} \frac{\kappa_{A^\alpha}}{a_\alpha}} t^{\sum_{\alpha=1}^{L} \frac{|A^\alpha|}{a_\alpha}} \cdot p_{\overrightarrow{A}}] \\
&= \frac{t^{\sum_{\alpha=1}^{L} \ell_\alpha}}{\mathfrak{z}_{\overrightarrow{\lambda}}} \cdot \mathrm{tr}(\beta_{\overrightarrow{n}}(\overrightarrow{w}) \cdot (\omega_{\lambda^1} \otimes \cdots \otimes \omega_{\lambda^L})) \\
&= \frac{t^{\sum_{\alpha=1}^{L} a_\alpha \ell_\alpha (w_\alpha a_\alpha + 1)}}{\mathfrak{z}_{\overrightarrow{\lambda}}} \cdot W_{(1)^{\ell(\overrightarrow{\lambda})}}(\mathcal{L}^{twist}_{\overrightarrow{n},\overrightarrow{w}}, q, t).
\end{aligned}
$$

As in the proof of Theorem 7.5, we get

$$F^{SO}_{\overrightarrow{\lambda}}(\mathcal{L};q,t,\overrightarrow{\tau}) = \frac{t^{\sum_{\alpha=1}^{L} a_\alpha \ell_\alpha (w_\alpha a_\alpha + 1)}}{\prod_{\alpha=1}^{L} \ell_\alpha ! a_\alpha^{\ell_\alpha}} \cdot F^{SO}_{(1)^{\ell(\overrightarrow{\lambda})}}(\mathcal{L}^{twist}_{\overrightarrow{n},\overrightarrow{w}};q,t). \tag{9.2}$$

In particular, we get the following proposition.

Proposition 9.1. *For a rectangular partition $\overrightarrow{\lambda}$ such that $\mu^\alpha = ({a_\alpha}^{\ell_\alpha})$, and any tube of integers $\overrightarrow{w} = (w_1, \cdots, w_L)$, we have*

$$(q-q^{-1})^{2-\ell(\overrightarrow{\lambda})} \cdot F^{SO}_{\overrightarrow{\lambda}}(\mathcal{L};q,t,\overrightarrow{\tau}) \in \mathbb{Q}[q-q^{-1}][t,t^{-1}]$$

for $\overrightarrow{\tau} = (w_1 + \frac{1}{a_1}, w_2 + \frac{1}{a_2}, \cdots, w_L + \frac{1}{a_L})$.

Consider the embedding $\mathbb{Q}(q)[t,t^{-1}] \hookrightarrow \mathbb{Q}[[T]]((u))$ via the changing of variables $q = e^u$ and $t = e^T$, we can expend the rational function $F^{SO}_{\overrightarrow{\lambda}}(\mathcal{L};q,t,\overrightarrow{\tau})$ into a formal power series in variables u and T

$$F^{SO}_{\overrightarrow{\lambda}}(\mathcal{L};e^u,e^T,\overrightarrow{\tau}) = \sum_{k=0}^{\infty} \sum_{i \geq -||\overrightarrow{\lambda}||} P_{k,i}(\overrightarrow{\tau}) T^k u^i$$

with coefficients $P_{k,i} \in \mathbb{Q}[\tau^1, \cdots, \tau^L]$.

The above proposition implies that the coefficients $P_{k,i}(\overrightarrow{\tau})$ for $i < \ell(\overrightarrow{\lambda}) - 2$ vanish when every $\tau_k - \frac{1}{a_k}$ takes arbitrary integer values, which is possible only when the polynomials $P_{k,i}(\overrightarrow{\tau})$ for $i < \ell(\overrightarrow{\lambda}) - 2$ are zero polynomials (a lattice is Zariski dense). Now specialize at the framing $\tau_1 = \tau_2 = \cdots = \tau_L = 0$ leads to the following theorem.

Proposition 9.2. *Suppose $\overrightarrow{\lambda}$ is a rectangular partition, the formal power series $F^{SO}_{\overrightarrow{\lambda}}(\mathcal{L};e^u,t)$ and $g_{\overrightarrow{\lambda}}(\mathcal{L};e^u,t)$ in the valuation field $\mathbb{Q}(t)((u))$ has u-valuation greater or equal to $\ell(\overrightarrow{\lambda}) - 2$.*

10. Appendix

10.1. **The Case of Unknot.** In this Appendix, we calculate the $F^{SO}(\bigcirc^L;q,t)$. We only deal with the case of unknot, i.e., $L = 1$, since the general case $L \geq 1$ can be done exactly the same way, except that the notation will be more complicated.

Proposition 10.1.

$$\sum_{A \in \widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) W^{SO}_A(\bigcirc;q,t)) = \prod_{i=1}^{\ell(\mu)} [1 + \frac{t^{\mu_i} - t^{-\mu_i}}{q^{\mu_i} - q^{-\mu_i}}].$$

Proof. Let $t = q^{2N}$ and compare with the quantum group definition of the colored Kauffman polynomials,

$$\begin{aligned}
&\sum_{A\in\widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) W_A^{SO}(\bigcirc; q, q^{2N}))\\
&= \sum_{A\in\widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) tr_{V_A}(K_{2\rho})\\
&= \sum_{A\in\widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) sb_A\\
&= pb_\mu(q^{1-2N}, q^{3-2N}, \cdots, q^{-1}, 1, q, \cdots, q^{2N-3}, q^{2N-1})\\
&= \prod_{i=1}^{\ell(\mu)} [1 + \frac{t^{\mu_i} - t^{-\mu_i}}{q^{\mu_i} - q^{-\mu_i}}].
\end{aligned}$$

As both the left and right hand side of the equation in the proposition are rational functions in t, and they agree for arbitrary sufficiently large N, they must coincide. ☐

Proposition 10.2.

$$Z_{CS}^{SO}(\bigcirc; q, t) = \exp(\sum_{k=1}^{+\infty} \frac{1}{k}(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}) pb_k). \tag{10.1}$$

$$\begin{aligned}
Z_{CS}^{SO}(\bigcirc; q, t) &= \sum_{\lambda\in\mathcal{P}} \frac{1}{\mathfrak{z}_\lambda} \cdot \prod_{i=1}^{\ell(\lambda)} [1 + \frac{t^{\lambda_i} - t^{-\lambda_i}}{q^{\lambda_i} - q^{-\lambda_i}}] \cdot pb_\lambda\\
&= \sum_{n=1}^{+\infty} \sum_{l(\lambda)=n} \frac{1}{\mathfrak{z}_\lambda} \cdot \prod_{i=1}^{\ell(\lambda)} [1 + \frac{t^{\lambda_i} - t^{-\lambda_i}}{q^{\lambda_i} - q^{-\lambda_i}}] \cdot pb_\lambda\\
&= \sum_{n=1}^{+\infty} \sum_{\lambda_1,\cdots,\lambda_n=1}^{+\infty} \frac{1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i} \prod_{i=1}^{n} [1 + \frac{t^{\lambda_i} - t^{-\lambda_i}}{q^{\lambda_i} - q^{-\lambda_i}}] \cdot pb_\lambda\\
&= \sum_{n=1}^{+\infty} \frac{1}{n!} [\sum_{k=1}^{+\infty} \frac{1}{k}(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}) pb_k]^n\\
&= \exp(\sum_{k=1}^{+\infty} \frac{1}{k}(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}) pb_k).
\end{aligned}$$

So we get the expression of free energy

$$F^{SO}(\bigcirc; q, t) = \sum_{k=1}^{+\infty} \frac{1}{k}(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}) pb_k.$$

Remark 10.1. This expression appeared in [6], but is computed from the path integral definition of the Chern-Simons partition function. Our derivation is based on our mathematical definition in terms of quantum group invariants and representations of Brauer algebra.

10.2. **An Alternative Definition of Colored Kauffman Polynomial via Markov Trace and Hopf Link.** The quantum group approach to the knot/link theory has produce a lot of invariants via the representation theory. However, the difficulty is that the calculation involved

are usually very complicated to compute. Fortunately, only quantum trace are essentially used. This enable us to find a combinatorial way instead of the quantum group method. Birman-Wenzl [5] and Wenzl [51] introduce a Markov trace definition. We will briefly introduce their construction here.

There is a well-defined *Markov trace* tr on the union of BMW algebra C_n with the following properties.

(1) $\mathrm{tr}(h(\alpha_1)h(\alpha_2)) = \mathrm{tr}(h(\alpha_2)h(\alpha_1))$ for any $\alpha_i \in B_n$
(2) $\mathrm{tr}(h(\beta)g_n^{\pm 1}) = \frac{t^{\pm 1}}{x}\mathrm{tr}(h(\beta))$ for any $\beta \in B_n$
(3) $\mathrm{tr}(1) = 1$
(4) $\mathrm{tr}(h(\beta)) = x^{1-n} t^{2\sum\limits_{i<j} lk_{ij}} K(\widehat{\beta}, q, t)$,
where $\widehat{\beta}$ is the link by closing the braid $\beta \in B_n$ and $K(\mathcal{L}, q, t)$ is the classic Kauffman polynomial of the link $\mathcal{L}$.

First normalize the trace by setting

$$\mathrm{Tr}(\xi) = x^n \cdot \mathrm{tr}(\xi) \text{ for } \xi \in C_n.$$

Let $\mathcal{L}$ be a link with L components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$, represented by the closure of $\beta \in B_m$. We associate each $\mathcal{K}_\alpha$ an irreducible representation V_{A^α} of quantized universal enveloping algebra $U_q(\mathfrak{so}(2N+1))$. Let $p_\alpha \in C_{d_\alpha}, \alpha = 1, \cdots, L$ be L minimal idempotents corresponding to the irreducible representations $V_{A^1}, \cdots, V_{A^L}$, where A^α denote the partition of $|A^\alpha| = d_\alpha$ labelling V_{A^α}. Denote $\overrightarrow{d} = (d_1, \cdots, d_L)$ and let $i_1, \cdots, i_m$ be integers such that $i_k = \alpha$ if the k-th strand of β belongs to the α-th component of $\mathcal{L}$. Let $\beta_{\overrightarrow{d}}$ be the cabling braid of β, replacing the k-th strand of β by d_{i_k} parallel ones. Then

$$W^{SO}_{\overrightarrow{A}}(\mathcal{L}; q, t) = q^{-\sum\limits_{\alpha=1}^{L} \kappa_{A^\alpha} w(\mathcal{K}_\alpha)} t^{-\sum\limits_{\alpha=1}^{L} |A^\alpha| w(\mathcal{K}_\alpha)} \cdot \mathrm{Tr}\left(h(\beta_{\overrightarrow{d}}) \cdot (p_{i_1} \otimes \cdots \otimes p_{i_m})\right). \tag{10.2}$$

Now we look at a concrete example to illustrate this method.

Let $\mathcal{L}$ be the Hopf link, represented by the braid $\beta = g_1^2$. Set $z = q - q^{-1}$. Easy to get $W^{SO}_{(1),(1)}(\mathcal{L}) = x(\frac{t-t^{-1}}{z} + 1 + z(t - t^{-1}))$. Let μ be a partition of 2, and let A be a partition of 2 or 0 labelling the irreducible representations of Brauer algebra Br_2. The character table reads $\chi_{(1,1)}(\gamma_{(2)}) = -1$ and $\chi_{(2)}(\gamma_{(2)}) = \chi_{(1,1)}(\gamma_{(1,1)}) = \chi_{(1,1)}(\gamma_{(1,1)}) = 1$. The representation labelled by $A = (2)$ is the trivial representation.

We want to compute $W^{SO}_{(1),(2)}(\mathcal{L})$. The minimal idempotents (studied by [4]) in C_2 are

$$p_{(2)} = (\frac{q^{-1} + g_2}{q + q^{-1}})(1 - x^{-1}e_2),\ p_{(1)^2} = (\frac{q - g_2}{q + q^{-1}})(1 - x^{-1}e_2) \text{ and } p_\phi = x^{-1}e_2,$$

where ϕ is the empty partition.

Denote the cabling of β by $\beta_{1,2}$, which is given by

$$h(\beta_{1,2}) = g_1 g_2^2 g_1.$$

By using the definition of BMW algebras C_n and the properties of Markov trace. We have the following formulas for the twisted cabling braids

$$\begin{aligned}
\mathrm{tr}(h(\beta_{1,2})\cdot g_2) &= \mathrm{tr}(g_1g_2^2g_1g_2)\\
&= \mathrm{tr}(g_1^4g_2)\\
&= \frac{t}{x}\mathrm{tr}(g_1^4)\\
&= \frac{t}{x}\mathrm{tr}((z+t^{-1})g_1^3+(1-t^{-1}z)g_1^2-t^{-1}g_1)\\
&= \frac{t}{x^2}((z+t^{-1})(2t-t^{-1}+(1-t^{-2})z+(t-t^{-1})z^2)\\
&+(1-t^{-1}z)(\frac{t-t^{-1}}{z}+1+z(t-t^{-1}))-1)\\
&= \frac{t}{x^2}(\frac{t-t^{-1}}{z}+1+(-2t^{-1}+3t-t^{-3})z+(1-t^{-2})z^2+(t-t^{-1})z^3),
\end{aligned}$$

where we used property $(P2)$ of BMW algebra C_n as well as the classic Kauffman polynomial of Trefoil knot and Hopf link.

Similarly, we have

$$\mathrm{tr}(h(\beta_{1,2}))=\frac{(x+zt-zt^{-1})^2}{x^2},$$

and

$$\mathrm{tr}(h(\beta_{1,2})\cdot e_2)=\frac{1}{x},$$

where $h(\beta_{1,2})\cdot e_2$ is actually the image of a link of the disjoint union of two unknots.

As

$$\begin{aligned}
p_{(2)}-p_{(1)^2}+p_\phi &= \frac{-z}{q+q^{-1}}+\frac{2g}{q+q^{-1}}+\frac{x^{-1}(z+q+q^{-1}-2g_2)e_2}{q+q^{-1}}\\
&= \frac{-z}{q+q^{-1}}+\frac{2g}{q+q^{-1}}+\frac{(z+q+q^{-1}-2t^{-1})e_2}{x(q+q^{-1})}
\end{aligned}$$

Then we have

$$\begin{aligned}
2Z_{(1),(2)}(\mathcal{L};q,t) &= W_{(1),(2)}(\mathcal{L};q,t)-W_{(1),(1,1)}(\mathcal{L};q,t)+W_{(1),(0)}(\mathcal{L};q,t)\\
&= x^3\mathrm{tr}[h(\beta_{1,2})\cdot(p_{(1)}\otimes(p_{(2)}-p_{(1)^2}+p_\phi))]\\
&= \frac{x^3}{q+q^{-1}}\cdot[2\mathrm{tr}(h(\beta_{1,2})\cdot g_2)-z\mathrm{tr}(h(\beta_{1,2}))+\frac{z+q+q^{-1}-2t^{-1}}{x}\mathrm{tr}(h(\beta_{1,2})\cdot e_2)]\\
&= \frac{x}{q+q^{-1}}[\frac{t^2-t^{-2}}{z}+(q+q^{-1})+(t^2-t^{-2})(z^3+4z)]
\end{aligned}$$

and

$$\begin{aligned}
2Z_{(1)}(\bigcirc)Z_{(2)}(\bigcirc) &= x(1+\frac{t^2-t^{-2}}{q^2-q^{-2}})\\
&= \frac{x}{q+q^{-1}}(q+q^{-1}+\frac{t^2-t^{-2}}{z}),
\end{aligned}$$

Thus we have

$$\begin{aligned}
2F_{(1),(2)}(\mathcal{L},q,t) &= 2Z_{(1),(2)}(\mathcal{L}) - 2Z_{(1)}(\bigcirc)Z_{(2)}(\bigcirc)\\
&= \frac{x}{(q+q^{-1})}(t^2-t^{-2})(z^3+4z)\\
&= (q+q^{-1})(t^2-t^{-2})[z+(t-t^{-1})]
\end{aligned}$$

and

$$\frac{2(q-q^{-1})^2F_{(1),(2)}}{(q-q^{-1})(q^2-q^{-2})} = (t^2-t^{-2})[(t-t^{-1})+z] \in \mathbb{Z}[z][t,t^{-1}]$$

as predicted in the Conjecture 5.1. Actually this example has already been discussed in Case C of Example 1 in Section 5.

10.3. **Explicit Computation of Quantum Trace for Orthogonal Quantum Group.** The universal matrix $\check{\mathcal{R}}$ acting on $V\otimes V$ for the natural representation of $U_q(\mathfrak{so}(2N+1))$ on V is given by Turaev [50]:

$$\begin{aligned}
\check{\mathcal{R}} =& q\sum_{i\neq N+1} E_{i,i}\otimes E_{i,i} + E_{N+1,N+1}\otimes E_{N+1,N+1} + \sum_{j}\sum_{\substack{i\neq j\\ i\neq 2N+2-j}} E_{j,i}\otimes E_{i,j}\\
&+ q^{-1}\sum_{i\neq N+1} E_{2N+2-i,i}\otimes E_{i,2N+2-i} + (q-q^{-1})\sum_{i<j} E_{i,i}\otimes E_{j,j}\\
&- (q-q^{-1})\sum_{i<j} q^{\bar{i}-\bar{j}} E_{2N+2-j,i}\otimes E_{j,2N+2-i},
\end{aligned}$$

where $E_{i,j}$ is the $(2N+1)\times(2N+1)$ matrix with

$$(E_{i,j})_{kl} = \begin{cases} 1 & (k,l)=(i,j)\\ 0 & elsewhere \end{cases}$$

and

$$\bar{i} = \begin{cases} i+\frac{1}{2} & 1\le i\le N\\ i & i=N+1\\ i-\frac{1}{2} & N+2\le i\le 2N+1 \end{cases}.$$

The enhancement of $\check{\mathcal{R}}$, $K_{2\rho}$ is given by

$$K_{2\rho}(v_i) = \begin{cases} q^{2i-1-2N}v_i & 1\le i\le N\\ v_i & i=N+1\\ q^{2i-3-2N}v_i & N+2\le i\le 2N+1 \end{cases}.$$

Then we compute the $\theta_V = \mathrm{tr}_V\check{\mathcal{R}}_{V,V}$ as follows

$$
\begin{aligned}
\mathrm{tr}_V \check{\mathcal{R}}_{V,V} &= q \sum_{i \neq N+1} \mathrm{tr}(E_{i,i} K_{2\rho}) \cdot E_{i,i} + \mathrm{tr}(E_{N+1,N+1} K_{2\rho}) \cdot E_{N+1,N+1} \\
&\quad + \sum_{j} \sum_{\substack{i \neq j \\ i \neq 2N+2-j}} \mathrm{tr}(E_{i,j} K_{2\rho}) \cdot E_{j,i} + q^{-1} \sum_{i \neq N+1} \mathrm{tr}(E_{i,2N+2-i} K_{2\rho}) \cdot E_{2N+2-i,i} \\
&\quad + (q - q^{-1}) \sum_{i<j} \mathrm{tr}(E_{j,j} K_{2\rho}) \cdot E_{i,i} - (q - q^{-1}) \sum_{i<j} q^{\bar{i} - \bar{j}} \mathrm{tr}(E_{j,2N+2-i} K_{2\rho}) \cdot E_{2N+2-j,i} \\
&= q \left(\sum_{i=1}^{N} q^{2i-1-2N} E_{i,i} + \sum_{i=N+2}^{2N+1} q^{2i-3-2N} E_{i,i} \right) + E_{N+1,N+1} \\
&\quad + (q - q^{-1}) \left(\sum_{j=1}^{N} \sum_{i=1}^{j-1} q^{2j-1-2N} E_{i,i} + \sum_{i=1}^{N} E_{i,i} + \sum_{j=N+2}^{2N+1} \sum_{i=1}^{j-1} q^{2j-3-2N} E_{i,i} \right) \\
&\quad - (q - q^{-1}) \sum_{j=N+2}^{2N+2} q^{\overline{2N+2-j} - \bar{j}} q^{2j-3-2N} E_{2N+2-j,2N+2-j}.
\end{aligned}
$$

Then we need to check the action of $\mathrm{tr}_V \check{\mathcal{R}}_{V,V}$ on those basis v_i's.

(1) For $1 \leq k \leq N$, we have

$$
\begin{aligned}
\mathrm{tr}_V \check{\mathcal{R}}_{V,V}(v_k) &= q \cdot q^{2k-1-2N} v_k + (q - q^{-1}) \left(\sum_{j=k+1}^{N} q^{2j-1-2N} v_k + v_k + \sum_{j=N+2}^{2N+1} q^{2j-3-2N} v_k \right) \\
&\quad - (q - q^{-1}) q^{\bar{k} - \overline{2N+2-k}} q^{2(2N+2-k)-3-2N} v_k \\
&= (q^{2k-2N} + (q - q^{-1})(q^{2k+1-2N} \frac{q^{2N-2k} - 1}{q^2 - 1} + 1 + q \frac{q^{2N} - 1}{q^2 - 1} \\
&\quad - q^{k+\frac{1}{2}-(2N+2-k-\frac{1}{2})} q^{2N+1-2k})) v_k \\
&= q^{2N} v_k.
\end{aligned}
$$

(2) For $k = N+1$, we have

$$
\begin{aligned}
\mathrm{tr}_V \check{\mathcal{R}}_{V,V}(v_{N+1}) &= v_{N+1} + (q - q^{-1}) \left(\sum_{j=N+2}^{2N+1} q^{2j-3-2N} v_{N+1} \right) \\
&= (1 + (q - q^{-1})(q \frac{q^{2N} - 1}{q^2 - 1})) v_{N+1} \\
&= q^{2N} v_{N+1}.
\end{aligned}
$$

(3) For $N+2 \leq k \leq 2N+1$, we have

$$
\begin{aligned}
\mathrm{tr}_V \check{\mathcal{R}}_{V,V}(v_k) &= q \cdot q^{2k-3-2N} v_k + (q - q^{-1}) \left(\sum_{j=k+1}^{2N+1} q^{2j-3-2N} v_k \right) \\
&= (q^{2k-2-2N} + (q - q^{-1})(q^{2k-1-2N} \frac{q^{4N+2-2k} - 1}{q^2 - 1})) v_k \\
&= q^{2N} v_k.
\end{aligned}
$$

Thus we get the following result

$$\theta_V = q^{2N}\mathrm{id}_V.$$

10.4. **Character tables of Brauer Algebras and Type-B Schur Functions.** Here are some character tables for Brauer algebras. Write $pb_\lambda = \sum_{A\in\widehat{Br}_{|\lambda|}} \chi_A(\gamma_\lambda)sb_A$, and we compute the character table by the following formula proved by A. Ram as Theorem 5.1 in [41]:

$$\chi_\lambda(\gamma_\mu) = \sum_{\substack{\nu\vdash|\mu| \\ \nu\supset\lambda}} (\sum_{\beta even} c^\nu_{\lambda\beta})\chi_\nu^{S_{|\mu|}}(\gamma_\mu).$$

where $c^\nu_{\lambda\beta}$'s are called the Littlewood-Richardson coefficients and defined via type-A Schur functions as follows

$$s_\alpha s_\beta = \sum_{|\gamma|=|\alpha|+|\beta|} c^\gamma_{\alpha\beta}s_\gamma$$

Recall the character table for permutation group S_n

χ	(2)	$(1,1)$
(2)	1	1
$(1,1)$	-1	1

χ	(3)	$(2,1)$	$(1,1,1)$
(3)	1	1	1
$(2,1)$	-1	0	2
$(1,1,1)$	1	-1	1

χ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
(4)	1	1	1	1	1
$(3,1)$	-1	0	-1	1	3
$(2,2)$	0	-1	2	0	2
$(2,1,1)$	1	0	-1	-1	3
$(1,1,1,1)$	-1	1	1	-1	1

Combine the above formulas, we have the following tables of Brauer algebra for small partitions

χ	(2)	$(1,1)$
(0)	1	1

χ	(3)	$(2,1)$	$(1,1,1)$
(1)	0	1	3

χ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
(2)	0	0	2	2	6
$(1,1)$	0	0	-2	0	6
(0)	1	0	3	1	3

We thus obtain the following expressions

$$\begin{aligned}
&pb_{(1)} = sb_{(1)}\\
&pb_{(2)} = sb_{(2)} - sb_{(1,1)} + 1\\
&pb_{(1,1)} = sb_{(2)} + sb_{(1,1)} + 1\\
&pb_{(3)} = sb_{(3)} - sb_{(2,1)} + sb_{(1,1,1)}\\
&pb_{(2,1)} = sb_{(3)} - sb_{(1,1,1)} + sb_{(1)}\\
&pb_{(1,1,1)} = sb_{(3)} + 2sb_{(2,1)} + sb_{(1,1,1)} + 3sb_{(1)}\\
&pb_{(4)} = sb_{(4)} - sb_{(3,1)} + sb_{(2,1,1)} - sb_{(1,1,1,1)} + 1\\
&pb_{(3,1)} = sb_{(4)} - sb_{(2,2)} + sb_{(1,1,1,1)}\\
&pb_{(2,2)} = sb_{(4)} - sb_{(3,1)} + 2sb_{(2,2)} - sb_{(2,,1,1)} + sb_{(1,1,1,1)} + 2sb_{(2)} - 2sb_{(1,1)} + 3\\
&pb_{(2,1,1)} = sb_{(4)} + sb_{(3,1)} - sb_{(2,1,1)} - sb_{(1,1,1,1)} + 2sb_{(2)} + 1\\
&pb_{(1,1,1,1)} = sb_{(4)} + 3sb_{(3,1)} + 2sb_{(2,2)} + 3sb_{(2,1,1)} + sb_{(1,1,1,1)} + 6sb_{(2)} + 6sb_{(1,1)} + 3,
\end{aligned}$$

and conversely

$$\begin{aligned}
&sb_{(1)} = pb_{(1)}\\
&sb_{(2)} = \frac{1}{2}[pb_{(2)} + pb_{(1,1)}] - 1\\
&sb_{(1,1)} = \frac{1}{2}[-pb_{(2)} + pb_{(1,1)}]\\
&sb_{(3)} = \frac{1}{6}[2pb_{(3)} + 3pb_{(2,1)} + pb_{(1,1,1)}] - pb_{(1)}\\
&sb_{(2,1)} = \frac{1}{3}[-pb_{(3)} + pb_{(1,1,1)}] - pb_{(1)}\\
&sb_{(1,1,1)} = \frac{1}{6}[2pb_{(3)} - 3pb_{(2,1)} + pb_{(1,1,1)}]\\
&sb_{(4)} = \frac{1}{24}[6pb_{(4)} + 8pb_{(3,1)} + 3pb_{(2,2)} + 6pb_{(2,1,1)} + pb_{(1,1,1,1)}] - \frac{1}{2}[pb_{(2)} + pb_{(1,1)}]\\
&sb_{(3,1)} = \frac{1}{8}[-2pb_{(4)} - pb_{(2,2)} + 2pb_{(2,1,1)} + pb_{(1,1,1,1)}] - pb_{(1,1)} + 1\\
&sb_{(2,2)} = \frac{1}{12}[-4pb_{(3,1)} + 3pb_{(2,2)} + pb_{(1,1,1,1)}] - \frac{1}{2}[pb_{(2)} + pb_{(1,1)}]\\
&sb_{(2,1,1)} = \frac{1}{8}[2pb_{(4)} - pb_{(2,2)} - 2pb_{(2,1,1)} + pb_{(1,1,1,1)}] + \frac{1}{2}[pb_{(2)} - pb_{(1,1)}]\\
&sb_{(1,1,1,1)} = \frac{1}{24}[-6pb_{(4)} + 8pb_{(3,1)} + 3pb_{(2,2)} - 6pb_{(2,1,1)} + pb_{(1,1,1,1)}].
\end{aligned}$$

Some of the type-B Schur functions are listed as follows.

$$sb_{(1)} = 1 + \frac{t - t^{-1}}{q - q^{-1}}$$
$$sb_{(2)} = \left(1 + \frac{tq - t^{-1}q^{-1}}{q^2 - q^{-2}}\right) \frac{t - t^{-1}}{q - q^{-1}}$$
$$sb_{(1,1)} = \left(1 + \frac{tq^{-1} - t^{-1}q}{q^2 - q^{-2}}\right) \frac{t - t^{-1}}{q - q^{-1}}$$
$$sb_{(3)} = \left(1 + \frac{tq^2 - t^{-1}q^{-2}}{q^3 - q^{-3}}\right) \frac{t - t^{-1}}{q^2 - q^{-2}} \frac{tq - t^{-1}q^{-1}}{q - q^{-1}}$$
$$sb_{(2,1)} = \left(1 + \frac{t - t^{-1}}{q^3 - q^{-3}}\right) \frac{tq^{-1} - t^{-1}q}{q - q^{-1}} \frac{tq - t^{-1}q^{-1}}{q - q^{-1}}$$
$$sb_{(1,1,1)} = \left(1 + \frac{tq^{-2} - t^{-1}q^2}{q^3 - q^{-3}}\right) \frac{t - t^{-1}}{q^2 - q^{-2}} \frac{tq^{-1} - t^{-1}q}{q - q^{-1}}$$
$$sb_{(4)} = \left(1 + \frac{tq^3 - t^{-1}q^{-3}}{q^4 - q^{-4}}\right) \frac{t - t^{-1}}{q^3 - q^{-3}} \frac{tq - t^{-1}q^{-1}}{q^2 - q^{-2}} \frac{tq^2 - t^{-1}q^{-2}}{q - q^{-1}}$$
$$sb_{(3,1)} = \left(1 + \frac{tq - t^{-1}q^{-1}}{q^4 - q^{-4}}\right) \frac{tq^{-1} - t^{-1}q}{q^2 - q^{-2}} \frac{t - t^{-1}}{q - q^{-1}} \frac{tq^2 - t^{-1}q^{-2}}{q - q^{-1}}$$
$$sb_{(2,2)} = \left(1 + \frac{t - t^{-1}}{q^3 - q^{-3}}\right) \left(1 + \frac{t - t^{-1}}{q - q^{-1}}\right) \frac{tq^{-2} - t^{-1}q^2}{q^2 - q^{-2}} \frac{tq^2 - t^{-1}q^{-2}}{q^2 - q^{-2}}$$
$$sb_{(2,1,1)} = \left(1 + \frac{tq^{-1} - t^{-1}q}{q^4 - q^{-4}}\right) \frac{tq^{-2} - t^{-1}q^2}{q - q^{-1}} \frac{tq - t^{-1}q^{-1}}{q^2 - q^{-2}} \frac{t - t^{-1}}{q - q^{-1}}$$
$$sb_{(1,1,1,1)} = \left(1 + \frac{tq^{-3} - t^{-1}q^3}{q^4 - q^{-4}}\right) \frac{t - t^{-1}}{q^3 - q^{-3}} \frac{tq^{-1} - t^{-1}q}{q^2 - q^{-2}} \frac{tq^{-2} - t^{-1}q^2}{q - q^{-1}}$$

10.5. **Colored Kauffman Polynomials of Torus Links/Knots and Tables of Integer Coefficients** $N_{\overrightarrow{\mu},g,\beta}$**.** The torus link $\mathcal{L} = T(rL, kL)$ has L components if $(r, k) = 1$. We compute the orthogonal quantum group invariants by the following formula proved in Theorem 3.6.

$$W_{\vec{A}}^{SO}(\mathcal{L}; q, t) = q^{-kr\sum_{\alpha=1}^{L}\kappa_{A^\alpha}} t^{-k(r-1)n} \sum_{f=0}^{[\frac{rn}{2}]} \sum_{\lambda \vdash rn-2f} \widetilde{c}_{\overrightarrow{A}}^{\lambda} q^{\frac{k\kappa_\lambda}{r}} t^{-\frac{2fk}{r}} sb_\lambda(q, t) \tag{10.3}$$

The explicit formula for these type-B Schur functions $sb_\lambda(q, t)$ are computed in the above subsection.

Recall the definition of the constants $\widetilde{c}_{\overrightarrow{A}}^{\lambda}$ by the formula

$$\prod_{\alpha=1}^{L} sb_{A^\alpha}(z^r) = \sum_{f=0}^{[rn/2]} \sum_{\lambda \vdash rn-2f} \widetilde{c}_{\overrightarrow{A}}^{\lambda} sb_\lambda(z).$$

In the case $r = 2$ and $L = 1$, we have

$\widetilde{c}$	(0)	(2)	$(1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
(1)	1	1	-1	$\times$	$\times$	$\times$	$\times$	$\times$
(2)	1	1	-1	1	-1	1	0	0
$(1,1)$	1	1	-1	0	0	1	-1	1

;

in the case when $r = 1$ and $L = 2$, we have

$\widetilde{c}$	(0)	(1)	(2)	$(1,1)$	(3)	$(2,1)$	$(1,1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$(1),(1)$	1	$\times$	1	1	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$(2),(1)$	$\times$	1	$\times$	$\times$	1	1	0	$\times$	$\times$	$\times$	$\times$	$\times$
$(1,1),(1)$	$\times$	1	$\times$	$\times$	0	1	1	$\times$	$\times$	$\times$	$\times$	$\times$
$(2),(2)$	1	$\times$	1	1	$\times$	$\times$	$\times$	1	1	1	0	0
$(2),(1,1)$	0	$\times$	1	1	$\times$	$\times$	$\times$	0	1	0	1	0
$(1,1),(1,1)$	1	$\times$	1	1	$\times$	$\times$	$\times$	0	0	1	1	1
$(3),(1)$	0	$\times$	1	0	$\times$	$\times$	$\times$	1	1	0	0	0
$(2,1),(1)$	0	$\times$	1	1	$\times$	$\times$	$\times$	0	1	1	1	0
$(1,1,1),(1)$	0	$\times$	0	1	$\times$	$\times$	$\times$	0	0	0	1	1

;

in the case when $r = 1$ and $L = 3$, we have

$\widetilde{c}$	(0)	(1)	(2)	$(1,1)$	(3)	$(2,1)$	$(1,1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$(1),(1),(1)$	$\times$	3	$\times$	$\times$	1	2	1	$\times$	$\times$	$\times$	$\times$	$\times$
$(2),(1),(1)$	1	$\times$	3	2	$\times$	$\times$	$\times$	1	2	1	1	0
$(1,1),(1),(1)$	1	$\times$	2	3	$\times$	$\times$	$\times$	0	1	1	2	1

.

In this subsection, we provide the tables for the values of the integers $N_{\overrightarrow{\mu},g,\beta}$ in the following formula

$$\frac{\mathfrak{z}_{\overrightarrow{\mu}}(q-q^{-1})^2\cdot[g_{\overrightarrow{\mu}}(q,t)-g_{\overrightarrow{\mu}}(q,-t)]}{2\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha}-q^{-\mu_i^\alpha})}=\sum_{g\in\mathbb{Z}_+/2}\sum_{\beta\in\mathbb{Z}}N_{\overrightarrow{\mu},g,\beta}z^{2g}t^{\beta}.$$

Example 1: Take $r = 1$, the torus link $T(2, 2k)$ has 2 components. By (10.3), we have

$$\begin{aligned}
W_{(1)(1)} &= q^{2k}sb_{(2)} + q^{-2k}sb_{(1,1)} + t^{-2k}\\
W_{(2)(1)} &= q^{4k}sb_{(3)} + q^{-2k}sb_{(2,1)} + q^{-2k}t^{-2k}sb_{(1)}\\
W_{(1,1)(1)} &= q^{2k}sb_{(2,1)} + q^{-4k}sb_{(1,1,1)} + q^{2k}t^{-2k}sb_{(1)}\\
W_{(2)(2)} &= q^{8k}sb_{(4)} + sb_{(3,1)} + q^{-4k}sb_{(2,2)} + q^{-2k}t^{-2k}sb_{(2)} + q^{-6k}t^{-2k}sb_{(1,1)} + q^{-4k}t^{-4k}\\
W_{(2)(1,1)} &= q^{4k}sb_{(3,1)} + q^{-4k}sb_{(2,1,1)} + q^{2k}t^{-2k}sb_{(2)} + q^{-2k}t^{-2k}sb_{(1,1)}\\
W_{(1,1)(1,1)} &= q^{4k}sb_{(2,2)} + sb_{(2,1,1)} + q^{-8k}sb_{(1,1,1,1)} + q^{6k}t^{-2k}sb_{(2)} + q^{2k}t^{-2k}sb_{(1,1)} + q^{4k}t^{-4k}
\end{aligned}$$

$$\begin{aligned}
W_{(3)(1)} &= q^{6k}sb_{(4)} + q^{-2k}sb_{(3,1)} + q^{-4k}t^{-2k}sb_{(2)}\\
W_{(2,1)(1)} &= q^{4k}sb_{(3,1)} + sb_{(2,2)} + q^{-4k}sb_{(2,1,1)} + q^{2k}t^{-2k}sb_{(2)} + q^{-2k}t^{-2k}sb_{(1,1)}\\
W_{(1,1,1)(1)} &= q^{2k}sb_{(2,1,1)} + q^{-6k}sb_{(1,1,1,1)} + q^{4k}t^{-2k}sb_{(1,1)}
\end{aligned}$$

Case A: Torus link $T(2, 2k)$ with partitions $(1),(1)$
$N_{\overrightarrow{\mu},g,\beta} = 0$.
Case B: Torus link $T(2, 2k)$ with partitions $(1,1),(1)$

$k=1$	$\beta=-3$	-1	1	3
$g=0$	-1	3	-3	1

$k=2$	$\beta=-5$	-3	-1	1	3
$g=0$	-4	4	12	-20	8
1	-1	1	3	-9	6
2	0	0	0	-1	1

$k=3$	$\beta=-7$	-5	-3	-1	1	3
$g=0$	-9	9	-3	45	-72	30
1	-6	6	-1	39	-93	55
2	-1	1	0	11	-47	36
3	0	0	0	1	-11	10
4	0	0	0	0	-1	1

Case C: Torus link $T(2,2k)$ partitions $(2),(1)$

$k=1$	$\beta=-3$	-1	1	3
$g=0$	1	-1	-1	1

$k=2$	$\beta=-5$	-3	-1	1	3
$g=0$	2	-2	2	-6	4
1	1	-1	1	-5	4
2	0	0	0	-1	1

$k=3$	$\beta=-7$	-5	-3	-1	1	3
$g=0$	3	-3	-1	9	-18	10
1	4	-4	-1	15	-39	25
2	1	-1	0	7	-29	22
3	0	0	0	1	-9	8
4	0	0	0	0	-1	1

Case D: Torus link $T(2,2k)$ with partitions $(2),(2)$

$k=1$	$\beta=-3$	-1	1	3
$g=1/2$	-2	2	-2	2
$3/2$	-1	1	-1	1

$k=2$	$\beta=-5$	-3	-1	1	3
$g=1/2$	-8	4	20	-36	20
$3/2$	-24	20	40	-96	60
$5/2$	-22	21	29	-97	69
$7/2$	-8	8	9	-47	38
$9/2$	-1	1	1	-11	10
$11/2$	0	0	0	-1	1

Case E: Torus link $T(2,2k)$ with partitions $(3),(1)$

$k=1$	$\beta=-3$	-1	1	3
$g=1/2$	-1	0	0	1

$k=2$	$\beta=-5$	-3	-1	1	3
$g=1/2$	-4	4	0	-8	8
$3/2$	-5	5	0	-14	14
$5/2$	-1	1	0	-7	7
$7/2$	0	0	0	-1	1

Example 2: The torus knots $T(2,k)$, where k is an odd integer. Again we compute the following quantum group invariants by (10.3).

$$W_{(1)} = t^{-k}(q^k sb_{(2)}(q,t) - q^{-k} sb_{(1,1)}(q,t) + t^{-k})$$
$$W_{(2)} = t^{-2k}(q^{2k} sb_{(4)} - q^{-2k} sb_{(3,1)} + q^{-4k} sb_{(2,2)} + q^{-3k} t^{-k} sb_{(2)} - q^{-5k} t^{-k} sb_{(1,1)} + q^{-4k} t^{-2k})$$
$$W_{(1,1)} = t^{-2k}(q^{4k} sb_{(2,2)} - q^{2k} sb_{(2,1,1)} + q^{-2k} sb_{(1,1,1,1)} + q^{5k} t^{-k} sb_{(2)} - q^{3k} t^{-k} sb_{(1,1)} + q^{4k} t^{-2k})$$

Case A: Torus knot $T(2,k)$ with partition $(1,1)$

$k=3$	$\beta=-11$	-9	-7	-5	-3
$g=1/2$	36	-132	180	-108	24
3/2	105	-377	453	-207	26
5/2	112	-450	494	-165	9
7/2	54	-275	286	-66	1
9/2	12	-90	91	-13	0
11/2	1	-15	15	-1	0
13/2	0	-1	1	0	0

Case B: Torus knot $T(2,k)$ with partition (2)

$k=3$	$\beta=-11$	-9	-7	-5	-3
$g=1/2$	-6	26	-42	30	-8
3/2	-35	125	-161	85	-14
5/2	-56	210	-238	91	-7
7/2	-36	165	-174	46	-1
9/2	-10	66	-67	11	0
11/2	-1	13	-13	1	0
13/2	0	1	-1	0	0

Example 3: Take $r=1$, the torus link $T(3,3k)$ has 3 components. By (10.3), We have

$$\begin{aligned}
W_{(1),(1),(1)} &= q^{6k} sb_{(3)} + 2sb_{(2,1)} + q^{-6k} sb_{(1,1,1)} + 3t^{-2k} sb_{(1)} \\
W_{(2),(1),(1)} &= q^{10k} sb_{(4)} + 2q^{2k} sb_{(3,1)} + q^{-2k} sb_{(2,2)} + q^{-6k} sb_{(2,1,1)} \\
&\quad +3t^{-2k} sb_{(2)} + 2q^{-4k} t^{-2k} sb_{(1,1)} + q^{-2k} t^{-4k} \\
W_{(1,1),(1),(1)} &= q^{6k} sb_{(3,1)} + q^{2k} sb_{(2,2)} + 2q^{-2k} sb_{(2,1,1)} + q^{-10k} sb_{(1,1,1,1)} \\
&\quad +2q^{4k} t^{-2k} sb_{(2)} + 3t^{-2k} sb_{(1,1)} + q^{2k} t^{-4k}
\end{aligned}$$

Torus link $T(3,3k)$ with partitions $(2),(1),(1)$

$k=1$	$\beta=-3$	-1	1	3
$g=1/2$	2	-2	-10	10
3/2	0	0	-6	6
5/2	0	0	-1	1

$k=2$	$\beta=-5$	-3	-1	1	3
$g=1/2$	16	-48	176	-336	192
$3/2$	12	-68	452	-1036	640
$5/2$	2	-38	494	-1406	948
$7/2$	0	-10	286	-1056	780
$9/2$	0	-1	91	-467	377
$11/2$	0	0	15	-121	106
$13/2$	0	0	1	-17	16
$15/2$	0	0	0	-1	1

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